Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
J. Finan. Intermediation 18 (2009) 65–92
Journal of Financial Intermediation
www.elsevier.com/locate/jfi

Stress testing by financial intermediaries: Implications for portfolio selection and asset pricing

Gordon J. Alexander [a,*], Alexandre M. Baptista [b]

[a] *Department of Finance, University of Minnesota, Minneapolis, MN 55455, USA*
[b] *Department of Finance, The George Washington University, Washington, DC 20052, USA*

Received 5 December 2006

Available online 13 March 2008

Abstract

Financial intermediaries often use stress testing to set risk exposure limits. Accordingly, we examine a model with an agent who faces stress testing constraints and another who does not. Three results are obtained. First, when there are K^* binding constraints, the constrained agent's optimal portfolio exhibits $(K^* + 2)$-*fund separation*. Second, the effect of the constraints on the optimal portfolio is identical to that of an adjustment in the expected payoffs of the risky securities that tends to lower them. Third, a security's equilibrium expected return depends on *both* its systematic risk and its idiosyncratic returns in the states where the constraints bind.

JEL classification: G11; G12; G21; G28; D81

Keywords: Stress testing; Financial intermediaries; Portfolio selection; Asset pricing; Idiosyncratic returns; Risk management; Regulation

1. Introduction

Over the past twenty years, some investors have suffered sizeable losses due to extreme events. Examples of such events include the crash in the U.S. stock market of 1987 and the Asian crisis of 1997. More recently, the terrorist attacks in the U.S. (2001), Spain (2004), and the U.K. (2005) have tremendously affected financial markets.

* Corresponding author. Fax: +1 612 626 1335.
E-mail addresses: gjalex@umn.edu (G.J. Alexander), alexbapt@gwu.edu (A.M. Baptista).

1042-9573/$ – see front matter
doi:10.1016/j.jfi.2008.02.004

Since the occurrence of these events, the importance of risk management has been extensively recognized by financial intermediaries when deciding the amount of risk they are willing to bear. Furthermore, bank regulators now put an emphasis on risk management practices in attempting to reduce the fragility of banking systems.

Value-at-risk (VaR) and stress testing have emerged as two of the most popular risk management tools.[1] For example, under the Basle Capital Accord, banks use VaR in internal models to calculate minimum capital requirements associated with their exposures to market risk (see, e.g., Berkowitz and O'Brien, 2002; Jorion, 2002; and Alexander and Baptista, 2006a). Moreover, banks that use this approach must have in place a rigorous stress testing program (see Basle Committee on Banking Supervision, 2005, p. 42). Not surprisingly, stress testing is now often used by financial intermediaries to set risk exposure limits (see Committee on the Global Financial System, 2005, pp. 1, 15; and Scholes, 2000).

While the previous literature examines the impact of using VaR as a risk management tool on portfolio selection and asset pricing (see, e.g., Alexander and Baptista, 2002), it has yet to similarly explore the impact of using stress testing constraints.[2] We provide parsimonious characterizations of (1) optimal portfolios in the presence of stress testing constraints and (2) equilibrium security expected returns in a two-agent economy where one agent faces these constraints and the other does not.[3]

An examination of the impact of stress testing constraints is of interest for several reasons. First, as noted earlier, financial intermediaries often use them to set risk exposure limits. Second, stress testing constraints may bind even when a financial intermediary holds capital in excess of the minimum capital requirements as determined by the Basle Capital Accord. This can happen since stress testing captures extreme events where losses can be very large. Third, stress testing constraints can be utilized to control portfolio skewness and kurtosis. Finally, since the return distributions of managed portfolios can resemble those of certain options strategies (see, e.g., Merton, 1981; and Jagannathan and Korajczyk, 1986), the use of stress testing constraints can mitigate the option-like features in these distributions.

In investigating the impact of stress testing constraints, we use Markowitz's (1952, 1959) mean-variance model. There are important reasons for doing so. First, the mean-variance model is the cornerstone of portfolio theory. Second, it is widely used in practice to determine optimal asset allocations. Third, since a set of stress testing constraints captures some measure of risk beyond variance (i.e., returns in extreme events), a mean-variance objective function is of interest.[4] Fourth, the model has been extensively used in the banking literature (see, e.g., Hart

[1] For an examination of the effect of extreme events in the estimation of VaR, see Longin (2000).

[2] A stress testing constraint is a restriction on the set of portfolios that are available for selection that specifies: (1) the state used in stress testing and (2) the maximum loss admissible in this state.

[3] Rietz (1988) and Barro (2006) show that economic disasters may explain the equity premium puzzle of Mehra and Prescott (1985). Our paper differs from theirs in two respects. First, we explore the portfolio selection and asset pricing implications arising from an agent using stress testing constraints to control losses in extreme events, while they explore the asset pricing implications arising from an agent recognizing the possibility of economic disasters in the absence of such constraints. Second, we examine an economy where there are two agents with mean-variance objective functions, while they examine an economy where there is a representative agent with constant relative risk aversion.

[4] Previous work examines the asset pricing implications of preferences defined over measures of risk beyond variance. Our paper differs from this work in two respects. First, the importance of a measure of risk beyond variance arises in our model due to the existence of stress testing constraints while it arises in other models from agents having, for example, (1) a preference for right-skewed portfolios (e.g., Kraus and Litzenberger, 1976; and Harvey and Siddique, 2000), or (2) an aversion to kurtosis (e.g., Dittmar, 2002), or (3) an aversion to disappointment (e.g., Ang et al., 2006a). Second,

and Jaffee, 1974; Francis, 1978; Koehn and Santomero, 1980; Kim and Santomero, 1988; and Rochet, 1992).

We obtain three main results. First, the constrained agent's optimal portfolio exhibits (K^*+2)-*fund separation*, where K^* is the number of binding stress testing constraints. This result simplifies stress testing-constrained portfolio selection problems into (K^*+2)-fund allocation exercises that can be solved in two steps: (1) identifying the K^*+2 funds, and (2) allocating wealth among them. While two of these funds are also used by the unconstrained agent, the remaining K^* funds capture the need to satisfy the constraints.

Second, the effect of the constraints on the optimal portfolio is identical to that of an adjustment in the expected payoffs of the risky securities that tends to lower them.[5] As a result, the constraints tend to lead to an increase in the optimal holding of the risk-free security, thereby reducing an agent's exposure to risky securities.

Third, a security's equilibrium expected return is affected by *both* its systematic risk (i.e., beta) and its idiosyncratic returns in the states where the constraints bind.[6] Specifically, securities with negative (positive) idiosyncratic returns in these states have relatively high (low) expected returns in equilibrium. The intuition for this result is straightforward. Due to the constraints, it is costly (beneficial) for the constrained agent to hold securities with negative (positive) idiosyncratic returns in such states. This result provides further motivation to the literature in which security prices are not solely driven by systematic risk.

Previous theoretical papers in this literature recognize the importance of idiosyncratic risk.[7] For example, Levy (1978), Merton (1987), and Malkiel and Xu (2002) develop models where security expected returns depend on both systematic and idiosyncratic risk. The empirical literature reports evidence of the importance of idiosyncratic risk. Campbell et al. (2001) and Malkiel and Xu (2003) show that the risk of individual stocks has noticeably increased over time. Goyal and Santa-Clara (2003) provide evidence of a significant positive relation between (1) a measure of idiosyncratic risk given by the *equal*-weighted average stock risk and (2) the future stock market return. However, Bali et al. (2005) find no evidence of a significant relation between the *value*-weighted average stock risk and the future stock market return. Moreover, Guo and Savickas (2006) show that when the *value*-weighted average stock risk and aggregate stock market risk are jointly used to forecast the market return, there is a significant negative relation between

our model captures the effects of the returns in just those states used in the stress testing constraints on asset pricing while other models use the returns in either (i) all states (to capture the effects of skewness and kurtosis), or (ii) states where the level of wealth is below the certainty equivalent (to capture the effects of disappointment aversion).

[5] This result is related to the papers of Cvitanić and Karatzas (1992) and Jagannathan and Ma (2002) who show that the solution of a portfolio selection problem with constraints on security holdings (e.g., prohibition of short sales) coincides with the solution of a portfolio selection problem in the absence of these constraints if certain adjustments in the distribution of security returns are made. Our result differs from theirs in that we consider stress testing constraints rather than constraints on security holdings.

[6] A security's idiosyncratic return in a state is given by its return in this state *minus* its risk-adjusted return in the state. In our paper, the risk-adjustment is made using the CAPM of Sharpe (1964), Lintner (1965), and Mossin (1966).

[7] Here, idiosyncratic risk refers to the standard deviation of idiosyncratic returns computed with respect to a given asset pricing model (e.g., the CAPM or the Fama-French three-factor model). For example, when the CAPM is used, a security's idiosyncratic returns are given by the residuals of a regression of (a) the security's excess returns (over the risk-free rate) on (b) the market portfolio's excess returns. Hence, the notion of 'idiosyncratic returns in the states where the stress testing constraints bind' differs from that of 'idiosyncratic risk.' The former consists of a relatively small set of the aforementioned residuals (i.e., those associated with the states where the stress testing constraints bind). The latter refers to the standard deviation of all residuals. Hence, while the former is a set of possibly more than one value (when more than one stress testing constraint binds), the latter always refers to a single value.

the *value*-weighted average stock risk and the future stock market return. Ang et al. (2006b, in press) find that individual stocks with higher idiosyncratic risk earn lower average returns.

Two features of our work differ from the idiosyncratic risk literature. First, the reason why idiosyncratic returns affect security expected returns in our model (i.e., the stress testing constraints) is, to the best of our knowledge, novel. Second, our model captures the effects of idiosyncratic *returns* in some states (i.e., those where the stress testing constraints bind) on asset pricing while the models in the literature capture the effects of idiosyncratic *risk* on asset pricing. Under certain conditions, the model generates predictions of security expected returns that are consistent with the empirical evidence in Ang et al. (2006a, in press). For example, consider a security with (1) high idiosyncratic risk and (2) positive idiosyncratic returns in the states where the stress testing constraints bind. Consistent with empirical evidence, our model predicts that this security has a relatively low expected return. However, it is important to emphasize that since the use of stress testing constraints is a relatively new practice, stress testing does not fully explain the aforementioned empirical evidence. Nevertheless, the asset pricing implications of our results are likely to become increasingly more relevant since the usage of stress testing is expanding (see Committee on the Global Financial System, 2005, pp. 1, 3).

Also related to our work are papers that investigate portfolio selection when an agent keeps his or her wealth above a floor (see, e.g., Black and Perold, 1992; Grossman and Vila, 1992; Grossman and Zhou, 1993; Basak, 1995; and Cvitanić and Karatzas, 1995). Our work differs from these papers in two important ways. First, we examine the impact of a set of stress testing constraints rather than that of a floor.[8] Second, we use the mean-variance model rather than an expected utility continuous-time model.[9]

Some more recent but nevertheless extensive literature explores the portfolio selection and equilibrium implications arising from using a VaR constraint.[10] Several papers in this literature use a continuous-time expected utility model. Basak and Shapiro (2001) find that when an agent faces a VaR constraint at the initial date, he or she may select a larger exposure to risky assets than that in its absence. Yiu (2004) show that a dynamic VaR constraint leads an agent to reduce his or her exposure to risky assets. Cuoco and Liu (2006) find that the VaR-based capital requirements of the Basle Capital Accord are effective in curbing portfolio risk. Under certain conditions, Alexander and Baptista (2004, 2006a) show that adding a VaR constraint to the mean-variance model leads to the selection of portfolios with larger standard deviations than those of the portfolios selected without the VaR constraint. However, there are also conditions under which doing so leads to the selection of portfolios with smaller standard deviations (see Sentana, 2003; and Alexander and Baptista, 2004, 2006a). Daníelsson and Zigrand (2008) show that VaR-based regulation can lower systemic risk in an equilibrium model.[11]

[8] Note that a set of stress testing constraints differs from a floor (also referred to as maximum drawdown). While the former restricts the wealth of an agent in only the states where extreme events occur, the latter restricts it in all states. For an examination of the portfolio selection implications of adding a maximum drawdown constraint to the mean-variance model, see Alexander and Baptista (2006b).

[9] Nevertheless, Markowitz and van Dijk (2003) find that under certain conditions, the mean-variance model provides a good approximation to multi-period expected utility maximization.

[10] Alexander and Baptista (2008) examine the portfolio selection implications arising from a portfolio manager using a VaR constraint when he or she manages a portfolio relative to a benchmark as in Roll (1992). Basak et al. (2006, in press-b) explore the portfolio selection implications arising from adding a constraint involving tracking error (relative to a benchmark) to a manager's portfolio selection problem. These papers show that such constraints are beneficial from the perspective of investors.

[11] For an equilibrium model of financial contagion, see Kodres and Pritsker (2002).

The paper proceeds as follows. Section 2 describes our model and characterizes optimal portfolios and equilibrium security expected returns. Section 3 provides an example that illustrates our theoretical results and Section 4 concludes. Appendix A contains the proofs.

2. The model

Next, we describe the model.

2.1. Securities

Consider an economy with two dates ($t = 0, 1$). The uncertainty at date 1 is represented by S states ($s = 1, \dots, S$). Let $\boldsymbol{\pi} = [\pi_1 \ \cdots \ \pi_S]^\top$ be the vector of probabilities for these states with $\pi_s > 0$ denoting the probability of state s. There are $J \leqslant S - 1$ risky securities ($j = 1, \dots, J$) available for trade at date 0. The payoffs of these securities are given by a $J \times S$ matrix $\boldsymbol{D}$. The entry in row j and column s of this matrix is the payoff of security j in state s and is denoted by d_{js}. Let $\boldsymbol{d}_s \equiv [d_{1s} \ \cdots \ d_{Js}]^\top$ denote the vector of security payoffs in state s. The vector of expected date-1 security payoffs is given by $\bar{\boldsymbol{d}} = [\bar{d}_1 \ \cdots \ \bar{d}_J]^\top$ with $\bar{d}_j$ denoting security j's expected payoff. Let $\boldsymbol{V}$ be the $J \times J$ variance-covariance matrix of security date-1 payoffs with $V_{j_1 j_2}$ denoting the covariance between the payoffs of securities j_1 and j_2. As usual, suppose that $rank(\boldsymbol{V}) = J$. Each risky security is in positive net supply and has one share outstanding.

A risk-free security ($j = J + 1$) is also available for trade at date 0.[12] This security has a date-1 payoff of one in all states. The risk-free security is in zero net supply.

A portfolio is a vector $\boldsymbol{q} = [q_1 \ \cdots \ q_{J+1}]^\top$ with q_j denoting the holding of security j. A positive holding of a security represents a long position in the security, whereas a negative holding represents a short position. For each portfolio $\boldsymbol{q}$, let $\widehat{\boldsymbol{q}} \equiv [q_1 \ \cdots \ q_J]^\top$. Portfolio $\boldsymbol{q}$'s expected payoff is $\widehat{\boldsymbol{q}}^\top \bar{\boldsymbol{d}} + q_{J+1}$. The variance of portfolio $\boldsymbol{q}$'s payoff is $\widehat{\boldsymbol{q}}^\top \boldsymbol{V} \widehat{\boldsymbol{q}}$.

A vector of date-0 prices for the risky securities is given by $\boldsymbol{p} = [p_1 \ \cdots \ p_J]^\top$ with p_j denoting security j's price. For convenience, we use the date-0 price of the risk-free security as the numeraire.[13] That is, the date-0 price of the risk-free security is normalized to be one.[14] Fix a portfolio $\boldsymbol{q}$ with positive price. Portfolio $\boldsymbol{q}$'s expected return is $\frac{\widehat{\boldsymbol{q}}^\top \bar{\boldsymbol{d}} + q_{J+1}}{\widehat{\boldsymbol{q}}^\top \boldsymbol{p} + q_{J+1}} - 1$. The variance of portfolio $\boldsymbol{q}$'s return is $\frac{\widehat{\boldsymbol{q}}^\top \boldsymbol{V} \widehat{\boldsymbol{q}}}{(\widehat{\boldsymbol{q}}^\top \boldsymbol{p} + q_{J+1})^2}$.

2.2. Agents

There are two agents with a mean-variance objective function $U : \mathbb{R}^2_+ \to \mathbb{R}$ defined by

$$U\left(\overline{W}, \sigma_W^2\right) = \overline{W} - \frac{\gamma}{2} \sigma_W^2, \tag{1}$$

[12] Our results can be extended in a natural way to the case of absence of a risk-free security as in Black (1972).

[13] Researchers often use the price of the risk-free security as the numeraire; see, for example, Harris (1980) in the context of a general equilibrium model where agents have mean-variance objective functions.

[14] Doing so does not result in loss of generality. Setting the price of the risk-free security to one and appropriately adjusting the prices of the risky securities does not affect the set of portfolios that satisfy a budget constraint. The set of portfolios that satisfy a stress testing constraint also remains unaffected provided that the maximum loss admissible is appropriately adjusted.

where $\overline{W}$ and σ_W^2 denote, respectively, the expected value and variance of date-1 wealth, and $\gamma > 0$. The assumption that this function is linear in both expected value and variance of wealth allows us to derive parsimonious formulas for the effect of stress testing constraints on optimal portfolios and equilibrium security expected returns. More generally, mean-variance objective functions are commonly used by practitioners and researchers (see, e.g., Sharpe, 1987, pp. 33–39; and Grinold and Kahn, 2000, pp. 96–99).[15]

The *unconstrained* agent has $\gamma = \gamma^u$. The *constrained* agent has $\gamma = \gamma^c$, and is restricted to select a portfolio $\boldsymbol{q}$ that satisfies K stress testing constraints:

$$\frac{\widehat{\boldsymbol{q}}^\top \boldsymbol{d}_s + q_{J+1}}{\widehat{\boldsymbol{q}}^\top \boldsymbol{p} + q_{J+1}} - 1 \geqslant -T_s, \quad s = s_1, \ldots, s_K, \tag{2}$$

where (i) $1 \leqslant K \leqslant J - 1$, (ii) $s_1, \ldots, s_K$ are distinct states, and (iii) $T_{s_1}, \ldots, T_{s_K}$ are possibly distinct bounds in states $s_1, \ldots, s_K$, respectively.[16] Each of these states represents an 'extreme' event where at least some of the securities have particularly small payoffs.[17] We assume that all bounds are greater than zero. Since: (a) the bounds are assumed to be greater than zero and (b) the risk-free security has a return of zero in all states, a portfolio with a sufficiently large holding of the risk-free security satisfies the stress testing constraints. Hence, the set of portfolios satisfying the stress testing constraints is non-empty.

While in practice there is a wide range of tools used to stress test a portfolio, the constraints in Eq. (2) are formulated to specifically capture the use of scenario analysis in setting risk exposure limits.[18] Jorion (2007, p. 357) points out that scenario analysis consists of evaluating a portfolio under various extreme states of the world. These states of the world are captured in our model by the states used in Eq. (2), i.e., $s_1, \ldots, s_K$.

In practice, the scenarios that are analyzed can be based on either historical or hypothetical events. The Committee on the Global Financial System (2005, Table 3a) notes that widely used historical events include: (a) the crash in the U.S. stock market of 1987, (b) the Asian crisis of 1997, and (c) the terrorist attacks in the U.S. in September of 2001. These events can be captured in our model by specifying the appropriate security payoffs in the states used to represent the events.

[15] Berk (1997) gives joint conditions on utility functions and return distributions that lead to mean-variance objective functions. For arbitrary distributions, these objective functions can be motivated with a quadratic utility function (see Huang and Litzenberger, 1988, p. 61). When no distributional assumption is made, a mean-variance objective function can be used as an approximation to expected utility maximization (see Markowitz, 1987, pp. 52–70).

[16] Our results can also be extended to the case when there are several constrained agents with each one of them: (1) having a possibly different γ and (2) facing a possibly different set of stress testing constraints.

[17] Conditional Value-at-Risk (CVaR) also captures extreme events, but there are notable differences between risk management systems based on CVaR and stress testing. First, while a CVaR-based risk management system typically uses a single CVaR constraint, a stress testing-based risk management system typically uses several stress testing constraints. Second, the set of states used to assess whether a portfolio satisfies a CVaR constraint depends on the portfolio, but the state used to assess whether a portfolio satisfies a stress testing constraint does not. Third, while a CVaR constraint restricts the average loss conditional on the aforementioned set of states, a stress testing constraint restricts the loss in a given state; for an examination of the impact of a CVaR constraint in the mean-variance model, see Alexander and Baptista (2004, 2006a).

[18] For a description of these tools, see, e.g., Committee on the Global Financial System (2005) and Jorion (2007, Ch. 14). The stress testing constraints in our paper are intended to capture the use of stress testing at the level of an individual financial intermediary. For a survey of macro stress testing, where the fragility of the entire financial system is assessed, see Sorge (2004).

In regard to hypothetical events, the Committee on the Global Financial System (2005, Table 3a) notes the common use of, for example, U.S. or global economic outlooks and oil price scenarios. These events can be captured in our model by the states where the payoff of a certain portfolio (e.g., the market portfolio) is below an appropriate value.[19]

Note that the agent who faces the stress testing constraints can be viewed as representing the regulated players in the economy (e.g., banks). The agent who does not face the constraints can be viewed as representing the unregulated players in the economy (e.g., hedge funds) who provide portfolio insurance to the regulated players.[20] Alternatively, our model can be viewed as representing an economy where certain agents are 'pure' mean-variance optimizers, while others also optimize along the mean-variance criterion but have an aversion to low payoffs in the states used by the stress testing constraints.

The unconstrained agent's *initial holdings* of securities are given by $\boldsymbol{q}_u = [\boldsymbol{q}_{1u} \ \cdots \ \boldsymbol{q}_{J+1u}]^\top$ with q_{ju} denoting the holding of security j. Similarly, the constrained agent's initial holdings of securities are given by $\boldsymbol{q}_c = [\boldsymbol{q}_{1c} \ \cdots \ \boldsymbol{q}_{J+1c}]^\top$ with q_{jc} denoting the holding of security j. Since each risky security has one share outstanding and the risk-free security is in zero net supply, we have $\widehat{\boldsymbol{q}}_u + \widehat{\boldsymbol{q}}_c = \mathbf{1}$, where $\mathbf{1}$ is the J-dimensional vector $[1 \ \cdots \ 1]^\top$, and $q_{J+1u} + q_{J+1c} = 0$. Thus, an economy is summarized by $[(\boldsymbol{D}, \boldsymbol{\pi}), (\gamma_u, \gamma_c), (\boldsymbol{q}_u, \boldsymbol{q}_c), \{(s_k, T_{s_k})\}_{k=1}^K]$.

2.3. *Portfolio selection implications*

Next, we characterize the agents' optimal portfolios.

2.3.1. *Unconstrained agent's optimal portfolio*

A portfolio with positive price is on the *mean-variance boundary* if there is no other portfolio with: (a) the same price, (b) the same expected payoff, and (c) a smaller payoff variance. In Appendix A, we show that for any portfolio $\boldsymbol{q}$ on the mean-variance boundary, we have

$$\widehat{\boldsymbol{q}} = \alpha \boldsymbol{V}^{-1}(\bar{\boldsymbol{d}} - \boldsymbol{p}) \tag{3}$$

for some $\alpha \in \mathbb{R}$. A portfolio with positive price is *efficient* if it lies on this boundary and has an expected return equal to or greater than zero. Otherwise, the portfolio is *inefficient*.

We are interested in the case when $\boldsymbol{p}^\top \boldsymbol{V}^{-1}(\bar{\boldsymbol{d}} - \boldsymbol{p}) \neq 0$. Portfolio $\boldsymbol{q}_t \equiv [\frac{[\boldsymbol{V}^{-1}(\bar{\boldsymbol{d}}-\boldsymbol{p})]^\top}{\boldsymbol{p}^\top \boldsymbol{V}^{-1}(\bar{\boldsymbol{d}}-\boldsymbol{p})} \quad 0]^\top$ is referred to as the *tangency portfolio*. Portfolio $\boldsymbol{q}_f \equiv [\mathbf{0}^\top \ 1]^\top$, where $\mathbf{0}$ is the J-dimensional vector $[0 \ \cdots \ 0]^\top$, is referred to as the *risk-free portfolio*.

The unconstrained agent's optimal portfolio $\boldsymbol{q}_u^*$ solves

$$\max_{\boldsymbol{q} \in \mathbb{R}^{J+1}} \widehat{\boldsymbol{q}}^\top \bar{\boldsymbol{d}} + q_{J+1} - \frac{\gamma_u}{2} \widehat{\boldsymbol{q}}^\top \boldsymbol{V} \widehat{\boldsymbol{q}} \tag{4}$$

$$\text{s.t.} \quad \widehat{\boldsymbol{q}}^\top \boldsymbol{p} + q_{J+1} \leqslant \widehat{\boldsymbol{q}}_u^\top \boldsymbol{p} + q_{J+1u}. \tag{5}$$

Note that Eq. (5) is a budget constraint. The following result characterizes $\boldsymbol{q}_u^*$.

[19] Hence, the selection of the states used by the stress testing constraints depends on security payoffs, but not on their endogenously determined prices.

[20] For an examination of whether hedge funds should be regulated, see Daníelsson et al. (2005).

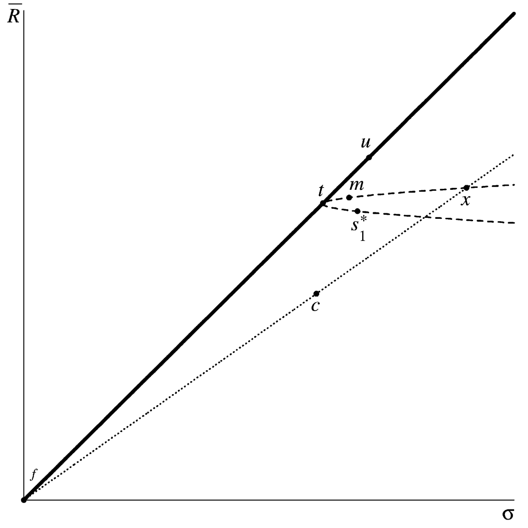

Fig. 1. Optimal portfolios and the market portfolio. *Note*. This figure illustrates the unconstrained and constrained agents' optimal portfolios ($\boldsymbol{q}_u^*$ and $\boldsymbol{q}_c^*$) and the market portfolio ($\boldsymbol{q}_m$), represented by, respectively, points u, c, and m. Also shown are funds $\boldsymbol{q}_f$, $\boldsymbol{q}_t$, $\boldsymbol{q}_{s_1^*}$, and $\boldsymbol{q}_x$, represented by, respectively, points f, t, s_1^*, and x. The line represents combinations of $\boldsymbol{q}_f$ and $\boldsymbol{q}_t$ (i.e., the efficient frontier). The dashed hyperbola represents combinations of $\boldsymbol{q}_t$ and $\boldsymbol{q}_{s_1^*}$. The dotted line represents combinations of $\boldsymbol{q}_f$ and $\boldsymbol{q}_x$.

Theorem 1. *The unconstrained agent's optimal portfolio is characterized by*

$$\boldsymbol{q}_u^* = \theta_f^u \boldsymbol{q}_f + \theta_t^u \boldsymbol{q}_t, \tag{6}$$

where $\theta_f^u \equiv (\widehat{\boldsymbol{q}}_u - \widehat{\boldsymbol{q}}_u^*)^\top \boldsymbol{p} + q_{J+1u}$ *and* $\theta_t^u \equiv \frac{\boldsymbol{p}^\top \boldsymbol{V}^{-1}(\overline{\boldsymbol{d}}-\boldsymbol{p})}{\gamma_u}$.

Theorem 1 says that $\boldsymbol{q}_u^*$ exhibits *two-fund separation*. Since these funds are $\boldsymbol{q}_f$ and $\boldsymbol{q}_t$, $\boldsymbol{q}_u^*$ is efficient. Note that if $\boldsymbol{p}^\top \boldsymbol{V}^{-1}(\overline{\boldsymbol{d}} - \boldsymbol{p})$ is positive, then $\boldsymbol{q}_t$ is efficient and θ_t^u is positive. However, if $\boldsymbol{p}^\top \boldsymbol{V}^{-1}(\overline{\boldsymbol{d}} - \boldsymbol{p})$ is negative, then $\boldsymbol{q}_t$ is inefficient and θ_t^u is negative. Fig. 1 illustrates Theorem 1 when $\theta_t^u > 0$. Note that $\boldsymbol{q}_f$, $\boldsymbol{q}_t$, and $\boldsymbol{q}_u^*$, represented by, respectively, points f, t, and u, lie on the line representing the efficient frontier.

2.3.2. Constrained agent's optimal portfolio

The constrained agent's optimal portfolio $\boldsymbol{q}_c^*$ solves

$$\max_{\boldsymbol{q} \in \mathbb{R}^{J+1}} \widehat{\boldsymbol{q}}^\top \overline{\boldsymbol{d}} + q_{J+1} - \frac{\gamma_c}{2} \widehat{\boldsymbol{q}}^\top \boldsymbol{V} \widehat{\boldsymbol{q}} \tag{7}$$

$$\text{s.t.} \quad \widehat{\boldsymbol{q}}^\top \boldsymbol{p} + q_{J+1} \leqslant \widehat{\boldsymbol{q}}_c^\top \boldsymbol{p} + q_{J+1c}, \tag{8}$$

$$\frac{\widehat{\boldsymbol{q}}^\top \boldsymbol{d}_s + q_{J+1}}{\widehat{\boldsymbol{q}}^\top \boldsymbol{p} + q_{J+1}} - 1 \geqslant -T_s, \quad s = s_1, \ldots, s_K. \tag{9}$$

Note that Eq. (8) is a budget constraint and that Eq. (9) represents the stress testing constraints. We are interested in the case when: (1) rank($[\bar{\boldsymbol{d}} - \boldsymbol{p} \quad \boldsymbol{d}_{s_1} - \boldsymbol{p} \quad \cdots \quad \boldsymbol{d}_{s_K} - \boldsymbol{p}]) = K + 1$;[21] and (2) $\boldsymbol{p}^\top \boldsymbol{V}^{-1}(\boldsymbol{d}_s - \boldsymbol{p}) \neq 0$ for $s = s_1, \ldots, s_K$. Let $\boldsymbol{q}_s \equiv [\frac{[\boldsymbol{V}^{-1}(\boldsymbol{d}_s - \boldsymbol{p})]^\top}{\boldsymbol{p}^\top \boldsymbol{V}^{-1}(\boldsymbol{d}_s - \boldsymbol{p})} \quad 0]^\top$ for $s = s_1, \ldots, s_K$.[22] Using Eq. (3), portfolios $\boldsymbol{q}_{s_1}, \ldots, \boldsymbol{q}_{s_K}$ are inefficient. These portfolios are useful to characterize $\boldsymbol{q}_c^*$.

Theorem 2. *The constrained agent's optimal portfolio is given by*

$$\boldsymbol{q}_c^* = \theta_f^c \boldsymbol{q}_f + \theta_t^c \boldsymbol{q}_t + \sum_{k=1}^{K^*} \theta_{s_k^*}^c \boldsymbol{q}_{s_k^*}, \tag{10}$$

where $\theta_f^c \equiv (\widehat{\boldsymbol{q}}_c - \widehat{\boldsymbol{q}}_c^*)^\top \boldsymbol{p} + q_{J+1c}$, $\theta_t^c \equiv \frac{\boldsymbol{p}^\top \boldsymbol{V}^{-1}(\bar{\boldsymbol{d}} - \boldsymbol{p})}{\gamma_c}$, $\theta_s^c \equiv \frac{\boldsymbol{p}^\top \boldsymbol{V}^{-1}(\boldsymbol{d}_s - \boldsymbol{p})}{\gamma_c} \lambda_s$ *for* $s = s_1^*, \ldots, s_{K^*}^*$, K^* *is the number of binding stress testing constraints,* $s_1^*, \ldots, s_{K^*}^*$ *are the states used by these constraints, and* $\lambda_{s_1^*}, \ldots, \lambda_{s_{K^*}^*}$ *are Lagrange multipliers associated with the constraints as given in Appendix A.*

Theorem 2 says that $\boldsymbol{q}_c^*$ exhibits $(K^* + 2)$-*fund separation*.[23] Since these funds are $\boldsymbol{q}_f$, $\boldsymbol{q}_t$, and $\boldsymbol{q}_{s_1^*}, \ldots, \boldsymbol{q}_{s_{K^*}^*}$ where the last K^* funds are inefficient, $\boldsymbol{q}_c^*$ is also inefficient. Fig. 1 illustrates Theorem 2 when $K^* = 1$ and $\theta_{s_1^*}^c < 0$. Note that $\boldsymbol{q}_{s_1^*}$ and $\boldsymbol{q}_c^*$, represented by, respectively, points s_1^* and c, lie below the efficient frontier. Using Eq. (10), we have

$$\boldsymbol{q}_c^* = \theta_f^c \boldsymbol{q}_f + \theta_x^c \boldsymbol{q}_x, \tag{11}$$

where $\theta_x^c \equiv \theta_t^c + \theta_{s_1^*}^c$ and $\boldsymbol{q}_x \equiv \frac{\theta_t^c \boldsymbol{q}_t + \theta_{s_1^*}^c \boldsymbol{q}_{s_1^*}}{\theta_x^c}$. Note that $\boldsymbol{q}_x$, represented by point x, lies on the dashed hyperbola representing combinations of $\boldsymbol{q}_t$ and $\boldsymbol{q}_{s_1^*}$. As Eq. (11) implies, $\boldsymbol{q}_c^*$ lies on the dotted line representing combinations of $\boldsymbol{q}_f$ and $\boldsymbol{q}_x$.

Of particular interest is the case when $\gamma_u = \gamma_c$. Using Theorems 1 and 2, we have $\theta_t^u = \theta_t^c$. Thus, the stress testing constraints do not affect the optimal holding of $\boldsymbol{q}_t$. The following result further explores the effect of the constraints on the optimal portfolio.

Theorem 3. *Suppose that* $\gamma_u = \gamma_c$ *and* $\boldsymbol{q}_u = \boldsymbol{q}_c$. *The constrained agent's optimal portfolio when the vector of security expected payoffs is* $\bar{\boldsymbol{d}}$ *coincides with the unconstrained agent's optimal portfolio when the vector of security expected payoffs is* $\bar{\boldsymbol{d}}^* \equiv \bar{\boldsymbol{d}} + \sum_{k=1}^{K^*} \lambda_{s_k^*}(\boldsymbol{d}_{s_k^*} - \boldsymbol{p})$, *where* K^* *is the number of binding stress testing constraints,* $s_1^*, \ldots, s_{K^*}^*$ *are the states used by these constraints, and* $\lambda_{s_1^*}, \ldots, \lambda_{s_{K^*}^*}$ *are Lagrange multipliers associated with the constraints as given in Appendix A.*

Using Theorem 3, the effect of the stress testing constraints on the optimal portfolio is identical to that of an adjustment in the vector of security expected payoffs $\bar{\boldsymbol{d}}$.[24] Since $\lambda_s > 0$ for

[21] This condition is not particularly restrictive. Given the security price vector, the condition holds for almost every security payoff matrix.

[22] It can be shown that portfolio $\boldsymbol{q}_s$ has minimum return variance among all portfolios with state-s returns equal to the state-s return of portfolio $\boldsymbol{q}_s$.

[23] Note that K^* depends on γ_c. For example, if γ_c is relatively small (large), then K^* might be equal to K (zero).

[24] Thus, the constrained agent's optimal portfolio coincides with that of an unconstrained agent with distorted beliefs regarding security payoffs. This result is related to the work of Maenhout (2004) where an agent cares about model

$s = s_1^*, \ldots, s_{K^*}^*$, security j's adjusted expected payoff $\bar{d}_j^*$ is smaller (larger) than its expected payoff $\bar{d}_j$ if d_{js} is smaller (larger) than p_j for $s = s_1^*, \ldots, s_{K^*}^*$.[25] In practice, a state is used by a stress testing constraint only if some securities have negative returns in the state. Hence, the case when $d_{js} \geqslant p_j$ for $j = 1, \ldots, J$ and $s = s_1^*, \ldots, s_{K^*}^*$ is not plausible.

In the case when $d_{js} < p_j$ for $j = 1, \ldots, J$ and $s = s_1^*, \ldots, s_{K^*}^*$, the expected payoffs of all risky securities are adjusted downward. Thus, the risk-free security is relatively more attractive in the presence of the constraints. Consequently, the optimal holding of this security in the presence of the constraints tends to be larger than that in their absence.

Consider now the case when $d_{j_1 s} < p_{j_1}$ and $d_{j_2 s} > p_{j_2}$ for some securities $j_1, j_2 \in \{1, \ldots, J\}$ and any state $s \in \{s_1^*, \ldots, s_{K^*}^*\}$. In this case, (1) the expected payoff of at least one risky security is adjusted downward and (2) the expected payoff of at least one risky security is adjusted upward. Thus, in the presence of stress testing constraints, the risk-free security can be more attractive, as attractive, or less attractive than in their absence. Consequently, the optimal holding of this security in the presence of the constraints can be smaller than, equal to, or larger than that in their absence.

2.4. Equilibrium implications

An *equilibrium* for economy $[(\boldsymbol{D}, \boldsymbol{\pi}), (\gamma_u, \gamma_c), (\boldsymbol{q}_u, \boldsymbol{q}_c), \{(s_k, T_{s_k})\}_{k=1}^K]$ is a vector of security prices $\boldsymbol{p}^*$ and a portfolio allocation $(\boldsymbol{q}_u^*, \boldsymbol{q}_c^*)$ such that: (a) given $\boldsymbol{p}^*$, $\boldsymbol{q}_u^*$ solves the unconstrained agent's portfolio selection problem; (b) given $\boldsymbol{p}^*$, $\boldsymbol{q}_c^*$ solves the constrained agent's portfolio selection problem; and (c) security markets clear, i.e.,

$$\widehat{\boldsymbol{q}}_u^* + \widehat{\boldsymbol{q}}_c^* = \mathbf{1}, \tag{12}$$

$$q_{J+1u}^* + q_{J+1c}^* = 0. \tag{13}$$

As Eqs. (12) and (13) indicate, in equilibrium the unconstrained agent absorbs the trades that the constrained agent is required to make to satisfy the stress testing constraints.

Let $\bar{R}_j$ and R_{js} denote, respectively, the expected return and state-s return of security j. Since all risky securities have one share outstanding, the market portfolio is given by $\boldsymbol{q}_m = [\mathbf{1}^\top\ 0]^\top$. Let $\bar{R}_m$ and R_{ms} denote, respectively, the expected return and state-s return of the market portfolio. Security j's beta is $\beta_j \equiv \sigma_{jm}/\sigma_m^2$, where σ_{jm} denotes the covariance between the returns on security j and the market portfolio and σ_m^2 denotes the variance of the market portfolio return.

We are interested in economies for which there exists an equilibrium where at least one of the stress testing constraints binds.[26] The following result characterizes security expected returns in equilibrium.

misspecification and seeks a portfolio rule that is robust (i.e., performs well not only when the underlying model is exact but also when it is misspecified). Consistent with max-min expected utility, a robust investor hedges against a worst-case model. Maenhout shows that robustness: (a) decreases the demand for risky assets and (b) is equivalent to an increase in risk aversion.

[25] Since the return on the risk-free security is zero, the condition that d_{js} is smaller (larger) than p_j is equivalent to the condition that security j's return in state s is smaller (larger) than the risk-free return.

[26] A characterization of the set of equilibria for a general class of economies with portfolio constraints is beyond the scope of our paper. For recent work on this issue, see Cass and Pavlova (2004), Basak et al. (in press-a), and references therein.

Theorem 4. *In equilibrium, the market portfolio is inefficient. Furthermore, security j's equilibrium expected return is given by*

$$\overline{R}_j = \beta_j \overline{R}_m - \sum_{k=1}^{K^*} \delta_{s_k^*}(R_{js_k^*} - \beta_j R_{ms_k^*}), \tag{14}$$

where $\delta_{s_k^*} \equiv \lambda_{s_k^*} \frac{1/\gamma_c}{1/\gamma_u + 1/\gamma_c}$ *is a positive constant for* $s = s_1^*, \ldots, s_{K^*}^*$, K^* *is the number of binding stress testing constraints,* $s_1^*, \ldots, s_{K^*}^*$ *are the states used by these constraints, and* $\lambda_{s_1^*}, \ldots, \lambda_{s_{K^*}^*}$ *are Lagrange multipliers associated with the constraints as given in Appendix A.*

Theorem 4 says that in equilibrium the market portfolio is inefficient. The intuition for this result is straightforward. Since (1) the unconstrained agent's optimal portfolio is efficient, (2) the constrained agent's optimal portfolio is inefficient, and (3) the market portfolio is the sum of these optimal portfolios, the market portfolio is inefficient. Fig. 1 illustrates that when $K^* = 1$, $\boldsymbol{q}_m$, represented by point m, lies below the efficient frontier on the dashed hyperbola representing combinations of $\boldsymbol{q}_t$ and $\boldsymbol{q}_{s_1^*}$.

Eq. (14) indicates that security j's equilibrium expected return depends on $K^* + 1$ terms. The first term, $\beta_j \overline{R}_m$, is identical to that in the CAPM when the risk-free return is zero (hereafter, 'the CAPM expected return').[27] However, the last K^* terms, which are subtracted from the first term, are not present in the CAPM. Each one of these K^* terms is given by the product of: (1) security j's *idiosyncratic return* in state s_k^*, $R_{js_k^*} - \beta_j R_{ms_k^*}$, and (2) the *risk premium* on the idiosyncratic return in state s_k^*, $\delta_{s_k^*}$, where $k \in \{1, \ldots, K^*\}$. Because the sum of these K^* terms is subtracted from the first term, we refer to $-\sum_{k=1}^{K^*} \delta_{s_k^*}(R_{js_k^*} - \beta_j R_{ms_k^*})$ as the *idiosyncratic return adjustment*.

Since $\delta_{s_k^*} > 0$ for $k = 1, \ldots, K^*$, whether security j's expected return is larger than, equal to, or smaller than that in the CAPM depends on its idiosyncratic returns in states $s_1^*, \ldots, s_{K^*}^*$. First, if security j's idiosyncratic return is zero in states $s_1^*, \ldots, s_{K^*}^*$, then its expected return is equal to that in the CAPM. Second, if security j's idiosyncratic return is (1) either zero or positive in states $s_1^*, \ldots, s_{K^*}^*$, and (2) positive for some state $s \in \{s_1^*, \ldots, s_{K^*}^*\}$, then its expected return is smaller than that in the CAPM. Third, if security j's idiosyncratic return is (1) either zero or negative in states $s_1^*, \ldots, s_{K^*}^*$, and (2) negative for some state $s \in \{s_1^*, \ldots, s_{K^*}^*\}$, then its expected return is larger than that in the CAPM. Finally, if security j's idiosyncratic return is negative in some state $s \in \{s_1^*, \ldots, s_{K^*}^*\}$ and positive in some state $s' \in \{s_1^*, \ldots, s_{K^*}^*\}$, then its expected return can be smaller than, equal to, or larger than that in the CAPM.

The intuition for why securities with negative (positive) idiosyncratic returns in states $s_1^*, \ldots, s_{K^*}^*$ have relatively high (low) expected returns in equilibrium is straightforward. Due to the constraints, it is costly (beneficial) for the constrained agent to hold securities with negative (positive) idiosyncratic returns in such states. Accordingly, the agent requires securities with negative (positive) idiosyncratic returns in these states to have relatively high (low) expected returns.

Theorem 4 implies that the *magnitude* of the effect of stress testing constraints on a security's expected return depends upon: (1) the size of the risk premia on the idiosyncratic returns in the states where the stress testing constraints bind and (2) the size of the security's idiosyncratic

[27] Comparisons between the equilibrium of Theorem 4 and the CAPM are made given the equilibrium security price vector. In general, however, this price vector in the former equilibrium will differ from that in the latter.

returns in these states. The magnitude of the effect is large when both: (1) the risk premia on the idiosyncratic returns in at least some of the states where the stress testing constraints bind are large; and (2) the security's idiosyncratic returns in all of these states are notably negative or positive. The magnitude of the effect is small when either: (1) the risk premia on the idiosyncratic returns in all states used by the binding stress testing constraints are small; or (2) the security's idiosyncratic returns in all of these states are close to zero. Hence, the magnitude of the effect of a set of stress testing constraints on the expected returns of different securities can be notably different.

3. Example

In this section, we illustrate our theoretical results using an example.

3.1. Securities

There are four risky securities ($j = 1, \ldots, 4$) in positive net supply. Each of these securities has one share outstanding. Panel (a) of Table 1 shows the expected values and standard deviations of their date-1 payoffs. For simplicity, assume that the correlation coefficient between the payoffs of each pair of distinct risky securities is 0.4.[28] There is also a risk-free security ($j = 5$) in zero net supply. This security has a risk-free date-1 payoff of one. We normalize the date-0 price of the risk-free security to be one. The length of time between dates 0 and 1 is assumed to be one month.

3.2. Agents

There are two agents: the unconstrained agent and the constrained agent. Both agents have the objective function given by Eq. (1) with $\gamma = \gamma_u = \gamma_c = 2$.[29] The constrained agent is restricted to select a portfolio that meets either one, two, or three stress testing constraints. Panel (b) of Table 1 shows the security payoffs in the states possibly used by these constraints, s_1, s_2, and s_3.[30] In selecting whether each of these states is used in stress testing, we proceed as follows. A state is used in stress testing if the payoff of the market portfolio in that state is below a certain value $C > 0$. Suppose that $C = 1.6$. Since the payoffs of the market portfolio in states s_1 and s_2 are, respectively, 1.487 and 1.536, both states are used in stress testing. However, since the payoff of the market portfolio in state s_3 is 1.624, this state is not used in stress testing. Thus, the constrained agent faces two stress testing constraints. Section 3.6 briefly examines the case when the constrained agent faces a single stress testing constraint, while Section 3.7 considers the case when he or she faces three stress testing constraints.

Note that the payoffs of all risky securities in state s_1 are smaller than those in state s_2. Thus, we examine stress testing constraints where the maximum loss that the constrained agent can suffer in state s_1 is larger than that he or she can suffer in state s_2. Specifically, we assume that the bounds are $T_{s_1} = 8\%$ and $T_{s_2} = 6\%$. While the values assumed for the bounds are used only

[28] Hence, the expected value of the market portfolio's payoff is 2, whereas its standard deviation is 0.11.

[29] Our results are robust to changes in γ.

[30] For brevity, we assume that the payoffs of the risky securities in other states are sufficiently high so that these states are not used in stress testing. Examples with similar results can be constructed where more than three stress testing constraints are used.

Table 1
Parameters used in the example
(a) Expected values and standard deviations of security payoffs

j	1	2	3	4	5
$\bar{d}_j$	0.200	0.400	0.600	0.800	1.000
$V_{jj}^{1/2}$	0.020	0.030	0.041	0.055	0.000

(b) Security payoffs in the states that are used in the stress testing constraints

j	1	2	3	4	5
d_{js_1}	0.136	0.352	0.455	0.544	1.000
d_{js_2}	0.145	0.370	0.474	0.547	1.000
d_{js_3}	0.149	0.372	0.551	0.552	1.000

(c) The agents' initial holdings of securities

j	1	2	3	4	5
q_{ju}	0.5	0.5	0.5	0.5	0.0
q_{jc}	0.5	0.5	0.5	0.5	0.0

Note. This table presents the parameters used in the example of Section 3. There are four risky securities ($j = 1, \ldots, 4$) and a risk-free security ($j = 5$). Panel (a) shows their expected payoffs ($\bar{d}_j$, $j = 1, \ldots, 5$) and standard deviations ($V_{jj}^{1/2}$, $j = 1, \ldots, 5$). Panel (b) shows the security payoffs in the states that are used in the stress testing constraints (d_{js}, $j = 1, \ldots, 5$, $s = s_1, s_2, s_3$). Panel (c) shows the unconstrained and constrained agents' initial holdings of securities ($\boldsymbol{q}_u$ and $\boldsymbol{q}_c$, respectively). Each risky security is in positive net supply and has one share outstanding. The risk-free security is in zero net supply. The length of time between dates 0 and 1 is assumed to be one month.

for illustrative purposes, Section 3.5 briefly examines the effect of using larger (smaller) bounds, which correspond to looser (tighter) stress testing constraints.

Panel (c) of Table 1 shows the agents' initial holdings of securities. In order to isolate the effect of the stress testing constraints in the optimal portfolio, the agents are assumed to have the same initial holdings of securities (and, as noted earlier, the same γ).

3.3. Portfolio selection implications

Next, we examine the agents' optimal portfolios. In doing so, we consider the equilibrium security prices provided in panel (a) of Table 2. Since the two stress testing constraints bind for the constrained agent, we have $s_1^* = s_1$ and $s_2^* = s_2$.

3.3.1. Funds required for optimal portfolios

Table 3 characterizes the funds required for the agents' optimal portfolios. Funds $\boldsymbol{q}_f$ and $\boldsymbol{q}_t$ are required for the two portfolios, but funds $\boldsymbol{q}_{s_1^*}$ and $\boldsymbol{q}_{s_2^*}$ are only required for the constrained agent's optimal portfolio. Panel (a) provides the security holdings of these four funds. The first row shows that (by definition) $\boldsymbol{q}_f$ has zero holdings of all risky securities and a unit holding of the risk-free security. The second row indicates that $\boldsymbol{q}_t$'s holdings range from 0.112 (security 2) to 0.653 (security 4). The relatively small holding of security 2 can be understood by the fact that this security has an expected return that is smaller than those of other risky securities as we show later. The third row indicates that $\boldsymbol{q}_{s_1^*}$ has positive holdings of securities 1, 3, and 4, and a negative holding of security 2. These results are driven by the state-s_1^* returns of the risky securities. Security 2's state s_1^*-return is $d_{2s_1^*}/p_2^* - 1 = -11.4\%$. In contrast, the state-s_1^* returns

Table 2
Equilibrium security prices, optimal portfolios, and adjusted security expected payoffs
(a) Security prices

j	1	2	3	4
p_j^*	0.198	0.397	0.595	0.792

(b) Optimal portfolios and the market portfolio

j	1	2	3	4	5
q_{ju}^*	0.918	0.167	0.790	0.971	−0.496
q_{jc}^*	0.082	0.833	0.210	0.029	0.496
q_{jm}	1.000	1.000	1.000	1.000	–

(c) Summary statistics on the returns of the optimal portfolios and the market portfolio

	$\overline{R}_{\boldsymbol{q}}$	$\sigma_{\boldsymbol{q}}$	$R_{\boldsymbol{q}s_1^*}$	$R_{\boldsymbol{q}s_2^*}$	$\rho_{\boldsymbol{q}}$
$\boldsymbol{q}_u^*$	17.7	29.9	−41.9	−39.0	–
$\boldsymbol{q}_c^*$	4.7	10.9	−8.0	−6.0	3.0
$\boldsymbol{q}_m$	11.2	19.3	−25.0	−22.5	0.4

(d) Adjustment in security expected payoffs

j	1	2	3	4
$\bar{d}_j$	0.200	0.400	0.600	0.800
$\lambda_{s_1^*}(d_{js_1^*} - p_j^*) + \lambda_{s_2^*}(d_{js_2^*} - p_j^*)$	−0.002	−0.001	−0.004	−0.007
$\bar{d}_j^*$	0.198	0.399	0.596	0.793

Note. Panel (a) shows the equilibrium security prices (p_j^*, $j = 1, \ldots, 4$) when the constrained agent faces two stress testing constraints that use states s_1 and s_2. The bounds are $T_{s_1} = 8\%$ and $T_{s_2} = 6\%$. Since both constraints bind for the constrained agent, we have $s_1^* = s_1$ and $s_2^* = s_2$. Panel (b) presents the unconstrained and constrained agents' optimal portfolios ($\boldsymbol{q}_u^*$ and $\boldsymbol{q}_c^*$, respectively) and the market portfolio ($\boldsymbol{q}_m$). Panel (c) provides summary statistics on the returns of the optimal portfolios and the market portfolio. Expected returns ($\overline{R}_{\boldsymbol{q}}$), standard deviations ($\sigma_{\boldsymbol{q}}$), and the distance from the mean-variance boundary ($\rho_{\boldsymbol{q}}$) are annualized, but the returns in states s_1^* and s_2^* ($R_{\boldsymbol{q}s_1^*}$ and $R_{\boldsymbol{q}s_2^*}$) are not. The numbers in this panel are reported in percentage points. Panel (d) provides the expected payoffs of the risky securities ($\bar{d}_j$, $j = 1, \ldots, 4$), the adjustments in their expected payoffs with the same effect on the optimal portfolio as that of the stress testing constraints, and the adjusted expected payoffs ($\bar{d}_j^*$, $j = 1, \ldots, 4$). These adjustments depend on: (i) the Lagrange multipliers associated with the constraints ($\lambda_{s_1^*}$ and $\lambda_{s_2^*}$), (ii) the payoffs of the risky securities in states s_1^* and s_2^* (d_{js}, $j = 1, \ldots, 4$, $s = s_1^*, s_2^*$), and (iii) the security prices.

of securities 1, 3, and 4 are, respectively, −31.3%, −23.5%, and −31.3%.[31] Hence, security 2 suffers a loss in state s_1^* that is notably smaller than those of securities 1, 3, and 4. The fact that $\boldsymbol{q}_{s_1^*}$ has a positive (negative) holding of a security with a relatively large (small) loss in state s_1^* might seem surprising. As we show shortly, however, the constrained agent's optimal portfolio has a negative holding of $\boldsymbol{q}_{s_1^*}$. The fourth row indicates that the results for $\boldsymbol{q}_{s_2^*}$ are similar to those for $\boldsymbol{q}_{s_1^*}$.

Panel (b) provides summary statistics on fund returns. The first two columns indicate that $\boldsymbol{q}_{s_1^*}$ and $\boldsymbol{q}_{s_2^*}$ have larger expected returns and standard deviations than those of $\boldsymbol{q}_t$. However, the third column shows that the former funds also suffer larger losses in states s_1^* and s_2^* than those

[31] These three returns are notably negative, but nevertheless plausible. For example, the Nasdaq index declined by 27.1% and 22.9% in, respectively, October 1987 and November 2000.

Table 3
Funds required for optimal portfolios
(a) Composition of funds

j	1	2	3	4	5
q_{jf}	–	–	–	–	1.000
q_{jt}	0.618	0.112	0.531	0.653	–
$q_{js_1^*}$	0.878	−0.551	0.596	0.872	–
$q_{js_2^*}$	0.787	−0.872	0.564	1.080	–

(b) Summary statistics on fund returns

	$\overline{R}_{\boldsymbol{q}}$	$\sigma_{\boldsymbol{q}}$	$R_{\boldsymbol{q}s_1^*}$	$R_{\boldsymbol{q}s_2^*}$	$\rho_{\boldsymbol{q}}$
$\boldsymbol{q}_f$	0.0	–	0.0	0.0	–
$\boldsymbol{q}_t$	11.8	19.9	−27.9	−26.0	–
$\boldsymbol{q}_{s_1^*}$	12.9	22.6	−32.9	−31.7	0.9
$\boldsymbol{q}_{s_2^*}$	13.4	24.7	−35.6	−35.0	2.0

(c) Fund holdings of the optimal portfolios and the market portfolio

	θ_f	θ_t	$\theta_{s_1^*}$	$\theta_{s_2^*}$
$\boldsymbol{q}_u^*$	−0.496	1.486	–	–
$\boldsymbol{q}_c^*$	0.496	1.486	−0.618	−0.373
$\boldsymbol{q}_m$	–	2.972	−0.618	−0.373

Note. This table characterizes the funds required for unconstrained and constrained agents' optimal portfolios ($\boldsymbol{q}_u^*$ and $\boldsymbol{q}_c^*$, respectively). Funds $\boldsymbol{q}_f$ and $\boldsymbol{q}_t$ are required for both portfolios. Funds $\boldsymbol{q}_{s_1^*}$ and $\boldsymbol{q}_{s_2^*}$ are only required for the constrained agent's optimal portfolio. Panel (a) provides the composition of these funds. Panel (b) provides summary statistics on fund returns. Expected returns ($\overline{R}_{\boldsymbol{q}}$), standard deviations ($\sigma_{\boldsymbol{q}}$), and the distance from the mean-variance boundary ($\rho_{\boldsymbol{q}}$) are annualized, but the returns in states s_1^* and s_2^* ($R_{\boldsymbol{q}s_1^*}$ and $R_{\boldsymbol{q}s_2^*}$) are not. In this panel, the numbers are reported in percentage points. Panel (c) provides the fund holdings of the optimal portfolios and the market portfolio. The holdings of funds $\boldsymbol{q}_f$, $\boldsymbol{q}_t$, $\boldsymbol{q}_{s_1^*}$, and $\boldsymbol{q}_{s_2^*}$ are denoted by, respectively, θ_f, θ_t, $\theta_{s_1^*}$, and $\theta_{s_2^*}$.

suffered by the latter. Since $\boldsymbol{q}_{s_1^*}$ and $\boldsymbol{q}_{s_2^*}$ are inefficient, their distance from the mean-variance boundary is of interest. We measure portfolio $\boldsymbol{q}$'s distance from this boundary, denoted by $\rho_{\boldsymbol{q}}$, in terms of annualized standard deviation.[32] The last column indicates that $\rho_{\boldsymbol{q}_{s_1^*}} = 0.9\%$ and $\rho_{\boldsymbol{q}_{s_2^*}} = 2.0\%$.

3.3.2. *Unconstrained agent's optimal portfolio*

The first row of panel (b) of Table 2 shows the unconstrained agent's optimal portfolio $\boldsymbol{q}_u^*$. Note that the agent's optimal holdings of the risky securities are all positive. However, the optimal holdings of securities 1, 3, and 4 are notably larger than that of security 2. This result follows from the fact that $\boldsymbol{q}_t$'s holding of security 2 is small relative to those of the other risky securities as mentioned earlier (see the second row of Table 3(a)). The agent's optimal holding of the risk-free security is negative. Hence, as panel (c) of Table 3 shows, the unconstrained agent borrows at the risk-free rate in order to take a leveraged position in $\boldsymbol{q}_t$. The first row of panel (c) of Table 2 says that the expected return and standard deviation of the unconstrained agent's optimal portfolio

[32] Note that $\rho_{\boldsymbol{q}}$ is the increase in standard deviation arising from selecting $\boldsymbol{q}$ instead of a portfolio on the mean-variance boundary with the same expected return.

are, respectively, 17.7% and 29.9%. Moreover, its returns in states s_1^* and s_2^*, at −41.9% and −39.0%, respectively, are notably negative.

3.3.3. Constrained agent's optimal portfolio

The second row of panel (b) of Table 2 shows the constrained agent's optimal portfolio $\boldsymbol{q}_c^*$. Comparing the optimal security holdings of the unconstrained and constrained agents, two main results can be seen. First, the constrained agent's optimal holdings of securities 1, 3, and 4 are much smaller than those of the unconstrained agent, while that of security 2 is much larger. The intuition for this result is straightforward. Security 2 suffers losses in states s_1^* and s_2^* that are notably smaller than those of securities 1, 3, and 4. Thus, the stress testing constraints force a reduction of the holdings in securities 1, 3, and 4, and an increase in the holding of security 2 by the constrained agent. Panel (d) provides the adjustments in the expected payoffs of the risky securities that, when ignoring the stress testing constraints, have the same effect on the optimal portfolio as the constraints. Note that the expected payoffs of all securities are adjusted downward. However, the adjustment for security 2 is in absolute terms smaller than those for securities 1, 3, and 4. Since securities 1, 3, and 4 are now relatively less attractive than security 2, the constraints cause the optimal holdings of the former securities to decrease and that of the latter to increase.

Second, the first column of panel (c) of Table 3 shows that the constrained agent's optimal holding of the risk-free security is positive, while that of the unconstrained agent is negative. Since the risk-free security has returns in states s_1^* and s_2^* that are larger than those of the risky securities, a positive holding of the risk-free security is useful to meet the stress testing constraints. The fact that the risk-free security is now more attractive relative to the risky securities can be understood by recalling that the effect of the stress testing constraints on the optimal portfolio is identical to that of a downward adjustment in the expected payoffs of all risky securities.

The second row of panel (c) of Table 2 says that the expected return and standard deviation of the constrained agent's optimal portfolio are, respectively, 4.7% and 10.9%. Thus, they are notably smaller than those of the unconstrained agent's optimal portfolio. It can be seen that the returns of the constrained agent's optimal portfolio in states s_1^* and s_2^* meet the stress testing constraints. Note that its distance from the mean-variance boundary is $\rho_{\boldsymbol{q}_c^*} = 3.0\%$.[33]

The second row of panel (c) of Table 3 shows the constrained agent's optimal fund holdings. Comparing the optimal fund holdings of the unconstrained and constrained agents, two results can be seen. First, the agents have the same optimal holding of $\boldsymbol{q}_t$. This result follows from the application of Theorems 1 and 2 when $\gamma_u = \gamma_c$. Second, while the unconstrained agent's optimal holdings of $\boldsymbol{q}_{s_1^*}$ and $\boldsymbol{q}_{s_2^*}$ are equal to zero, those of the constrained agent are negative. Since $\boldsymbol{q}_{s_1^*}$ and $\boldsymbol{q}_{s_2^*}$ suffer large losses in states s_1^* and s_2^*, short positions in $\boldsymbol{q}_{s_1^*}$ and $\boldsymbol{q}_{s_2^*}$ are useful to meet the stress testing constraints.

3.4. Equilibrium implications

Next, we characterize the market portfolio and equilibrium security expected returns.

[33] While stress testing constraints can be useful for an agent who, for example, wishes to control kurtosis, they are (by construction) costly for agents with mean-variance objective functions. The cost of the constraints for these agents can be measured by the reduction in the annualized certainty equivalent return (CER) incurred by the agent who selects the constrained optimal portfolio instead of the unconstrained optimal portfolio. Since the CERs of these portfolios are, respectively, 3.5% and 8.9%, this cost (and thus the cost of 'protection') is 5.4%.

Table 4
Risk premia, security characteristics, and security expected returns

(a) Risk premia

$\overline{R}_m$	11.2
$\delta_{s_1^*}$	9.6
$\delta_{s_2^*}$	6.5

(b) Security betas and idiosyncratic returns

j	1	2	3	4
β_j	1.16	0.94	0.93	1.04
$R_{js_1^*} - \beta_j R_{ms_1^*}$	−2.23	12.14	−0.18	−5.40
$R_{js_2^*} - \beta_j R_{ms_2^*}$	−0.56	14.35	0.70	−7.58

(c) Security expected returns

j	1	2	3	4
$\beta_j \overline{R}_m$ (CAPM expected return)	13.02	10.55	10.46	11.61
$-(R_{js_1^*} - \beta_j R_{ms_1^*})\delta_{s_1^*}$	0.21	−1.16	0.02	0.52
$-(R_{js_2^*} - \beta_j R_{ms_2^*})\delta_{s_2^*}$	0.04	−0.93	−0.05	0.49
Idiosyncratic return adjustment	0.25	−2.10	−0.03	1.01
$\overline{R}_j$	13.27	8.46	10.44	12.62

Note. Panel (a) provides the risk premia. The risk premium on security betas is the expected return on the market portfolio $\overline{R}_m$. The risk premia on idiosyncratic returns in states s_1^* and s_2^* are denoted by $\delta_{s_1^*}$ and $\delta_{s_2^*}$, respectively. Panel (b) presents the security betas (β_j, $j = 1, \ldots, 4$) and their idiosyncratic returns in states s_1^* and s_2^* ($R_{js} - \beta_j R_{ms}$, $j = 1, \ldots, 4$, $s = s_1^*, s_2^*$). Panel (c) presents the security expected returns ($\overline{R}_j$, $j = 1, \ldots, 4$). With the exception of the idiosyncratic returns in states s_1^* and s_2^*, all numbers are annualized. With the exception of the betas, all numbers are reported in percentage points.

3.4.1. *Market portfolio*

Since all risky securities have one share outstanding, the third row of panel (b) of Table 2 shows that $\boldsymbol{q}_m$'s holding of each risky security is one. The third row of panel (c) of Table 2 reports that $\boldsymbol{q}_m$'s expected return and standard deviation are, respectively, 11.2% and 19.3%. Thus, they are smaller (larger) than those of $\boldsymbol{q}_u^*$ ($\boldsymbol{q}_c^*$). Moreover, $\boldsymbol{q}_m$'s returns in states s_1^* and s_2^* are smaller (larger) in absolute terms than those of $\boldsymbol{q}_u^*$ ($\boldsymbol{q}_c^*$). Finally, $\boldsymbol{q}_m$'s distance from the mean-variance boundary is only $\rho_{\boldsymbol{q}_m} = 0.4\%$. This distance is small since $\boldsymbol{q}_m$'s holdings of $\boldsymbol{q}_{s_1^*}$ and $\boldsymbol{q}_{s_2^*}$ are relatively small (in absolute terms) when compared to $\boldsymbol{q}_m$'s holding of $\boldsymbol{q}_t$ (see the third row of Table 3(c)).

3.4.2. *Security expected returns*

Panel (a) of Table 4 shows the risk premia. First, since the risk-free return is zero, the CAPM's risk premium is $\overline{R}_m = 11.2\%$.[34] Second, the risk premium of the idiosyncratic return in state s_1^* is $\delta_{s_1^*} = 9.6\%$ so that a security's expected return decreases by 9.6 basis points per percentage point of its idiosyncratic return in state s_1^*. Third, the risk premium of the idiosyncratic return in state s_2^* is $\delta_{s_2^*} = 6.5\%$ so that a security's expected return decreases by 6.5 basis points per percentage point of its idiosyncratic return in state s_2^*.

The first row of panel (b) shows that the security betas range from 0.93 (security 3) to 1.16 (security 1). The second and third rows provide their idiosyncratic returns in states s_1^* and s_2^*,

[34] The fact that this risk premium is sizeable can be understood by noting that the unconstrained agent is required to absorb relatively large trades made by the constrained agent as he or she makes adjustments to meet the stress testing constraints. Since the unconstrained agent is thus required to bear more risk, he or she is willing to hold the risky securities only if they provide relatively high expected returns.

respectively. Securities 1 and 4 have negative idiosyncratic returns in both states. Security 2 has positive idiosyncratic returns in both states. Security 3 has a negative (positive) idiosyncratic return in state s_1^* (s_2^*).

Panel (c) decomposes the security expected returns that are provided in the last row. The first row provides the CAPM expected return. The next three rows provide the two terms that are not present in the CAPM and their sum, i.e., the idiosyncratic return adjustment. Since securities 1 and 4 have negative idiosyncratic returns in states s_1^* and s_2^* as shown in panel (b), their idiosyncratic return adjustments are positive. The idiosyncratic return adjustment for security 1 (0.25%) is smaller than that for security 4 (1.01%) because the idiosyncratic returns of the former security are also smaller in absolute terms than those of the latter. Since security 2 has positive and large idiosyncratic returns in states s_1^* and s_2^*, its idiosyncratic return adjustment is notably negative (−2.10%). Lastly, since security 3 has a slightly negative (positive) idiosyncratic return in state s_1^* (s_2^*), its idiosyncratic return adjustment is close to zero. In sum, the idiosyncratic return adjustments can be notably negative, close to zero, or notably positive.

3.5. Using other values for the bounds

In order to examine the effect of the size of the stress testing bounds on equilibrium expected returns, two additional cases were considered. These cases involve having $T_{s_1} = 10\%$ and $T_{s_2} = 8\%$ so that the two stress testing constraints are loosened, and having $T_{s_1} = 6\%$ and $T_{s_2} = 4\%$ so that the two stress testing constraints are tightened. In either of the two cases, both constraints bind for the constrained agent. Hence, we have $s_1^* = s_1$ and $s_2^* = s_2$. Table 5 presents the results for these cases in panels (a) and (c), along with those from panel (c) of Table 4 that now reappears in panel (b) in order to make easier viewing of the effect of the size of the bounds on expected returns.

The effect of tightening the constraints can be examined by moving from panel (a) to panel (b) and, in turn, to panel (c) of Table 5. In doing so, three results are apparent. First, CAPM expected returns become larger (see the first row in each panel). Second, the idiosyncratic return adjustment becomes slightly smaller for security 1 while these adjustments for the other securities become slightly larger in absolute terms (see the fourth row in each panel). Hence, the idiosyncratic return adjustments for securities 1 and 3 (2 and 4) remain relatively small (large). Third, equilibrium security expected returns become larger (see the fifth row in each panel). This result can be understood as follows. With tighter constraints, the constrained agent selects a portfolio that is 'farther' from that selected by the unconstrained agent. Hence, the unconstrained agent is required to absorb 'larger' trades made by the constrained agent as he or she makes portfolio adjustments to meet the constraints. Since the unconstrained agent is thus required to bear more risk, he or she is willing to hold the risky securities only if they provide higher expected returns.

In sum, the effect of stress testing constraints on the expected returns of certain securities can be large for various values of the bounds. However, there are also securities for which this effect is small for these values of the bounds.

3.6. Using a single stress testing constraint

Suppose that $C = 1.5$. Since the payoffs of the market portfolio in states s_1, s_2, and s_3 are, respectively, 1.487, 1.536, and 1.624, only state s_1 is used in stress testing and thus the constrained

Table 5
Effect of the size of the stress testing bounds on security expected returns

(a) $T_{s_1^*} = 10\%$ and $T_{s_2^*} = 8\%$

j	1	2	3	4
$\beta_j \overline{R}_m$ (CAPM expected return)	12.54	10.17	10.08	11.18
$-(R_{js_1^*} - \beta_j R_{ms_1^*})\delta_{s_1^*}$	0.24	−1.33	0.02	0.59
$-(R_{js_2^*} - \beta_j R_{ms_2^*})\delta_{s_2^*}$	0.02	−0.46	−0.02	0.24
Idiosyncratic return adjustment	0.26	−1.78	0.00	0.83
$\overline{R}_j$	12.80	8.38	10.07	12.01

(b) $T_{s_1^*} = 8\%$ and $T_{s_2^*} = 6\%$

j	1	2	3	4
$\beta_j \overline{R}_m$ (CAPM expected return)	13.02	10.55	10.46	11.61
$-(R_{js_1^*} - \beta_j R_{ms_1^*})\delta_{s_1^*}$	0.21	−1.16	0.02	0.52
$-(R_{js_2^*} - \beta_j R_{ms_2^*})\delta_{s_2^*}$	0.04	−0.93	−0.05	0.49
Idiosyncratic return adjustment	0.25	−2.10	−0.03	1.01
$\overline{R}_j$	13.27	8.46	10.44	12.62

(c) $T_{s_1^*} = 6\%$ and $T_{s_2^*} = 4\%$

j	1	2	3	4
$\beta_j \overline{R}_m$ (CAPM expected return)	13.51	10.95	10.85	12.05
$-(R_{js_1^*} - \beta_j R_{ms_1^*})\delta_{s_1^*}$	0.18	−1.00	0.02	0.45
$-(R_{js_2^*} - \beta_j R_{ms_2^*})\delta_{s_2^*}$	0.05	−1.41	−0.07	0.74
Idiosyncratic return adjustment	0.23	−2.41	−0.05	1.19
$\overline{R}_j$	13.74	8.54	10.80	13.24

Note. This table shows the effect of the size of the stress testing bounds on security expected returns. There are two stress testing constraints that use states s_1 and s_2. In panel (a), the bounds are $T_{s_1} = 10\%$ and $T_{s_2} = 8\%$. In panel (b), the bounds are $T_{s_1} = 8\%$ and $T_{s_2} = 6\%$. In panel (c), the bounds are $T_{s_1} = 6\%$ and $T_{s_2} = 4\%$. In all cases of the bounds, both constraints bind for the constrained agent. Hence, we have $s_1^* = s_1$ and $s_2^* = s_2$. The rest of the notation is defined as in Table 4. All numbers are annualized and reported in percentage points.

Table 6
Effect of a single stress testing constraint on security expected returns

j	1	2	3	4
$\beta_j \overline{R}_m$ (CAPM expected return)	13.14	10.65	10.56	11.72
$-(R_{js_1^*} - \beta_j R_{ms_1^*})\delta_{s_1^*}$ (Idiosyncratic return adjustment)	0.35	−1.93	0.03	0.86
$\overline{R}_j$	13.49	8.72	10.59	12.57

Note. This table presents the effect of a single stress testing constraint on security expected returns. This constraint uses state s_1 and bound $T_{s_1} = 8\%$. Since the constraint binds for the constrained agent, we have $s_1^* = s_1$. The rest of the notation is defined as in Table 4. All numbers are annualized and reported in percentage points.

agent faces only a single stress testing constraint. Assume that $T_{s_1} = 8\%$. Since the stress testing constraint binds for the constrained agent, we have $s_1^* = s_1$. Table 6 presents equilibrium security expected returns. Comparing the case of a single stress testing constraint with the case of two stress testing constraints with bounds $T_{s_1^*} = 8\%$ and $T_{s_2^*} = 6\%$ as shown in Table 5(b), three results can be seen. First, the CAPM expected returns of all risky securities in Table 6 are slightly larger than those in Table 5(b). Second, the idiosyncratic return adjustments in Table 6 are within 17 basis points of those in Table 5(b). Third, the security expected returns in Table 6 are within 26 basis points of those in Table 5(b). In sum, the effect of stress testing constraints on the expected returns of certain securities can be large regardless of whether one or two constraints are used. However, there are also securities for which this effect is small regardless of whether one or two constraints are used.

3.7. Using stress testing when there is a breakdown in correlations

There is an extensive literature examining whether correlations between the returns of different assets change under various market conditions. For example, King and Wadhwani (1990) provide evidence of an increase in the correlation between the U.S. and U.K. equity markets during the crash in the U.S. stock market of 1987. Longin and Solnik (2001) show that there is also an increase in the correlation between U.S. and international equity markets during bear periods. Chakrabarti and Roll (2002) find that stock market correlations within East Asia and within Europe increased during the 1997 Asian crisis.[35] For brevity, we refer to these changes in correlations as 'breakdown in correlations.' While a detailed examination of the portfolio selection and asset pricing implications of breakdowns in correlations is beyond the scope of our paper, we next illustrate the use of stress testing constraints when there is an increase in correlations and its effect on security expected returns.[36]

Suppose that there is a breakdown in correlations in states s_1, s_2, and s_3.[37] Accordingly, assume that the constrained agent faces three stress testing constraints that use such states.[38] For simplicity, suppose that the states are equally likely. Panel (a) of Table 7 shows the correlation coefficients between asset payoffs conditional on these states. For brevity, we refer to these coefficients as 'conditional correlation coefficients.'[39] As noted earlier, the (unconditional) correlation coefficient between the payoffs of each pair of different risky assets is assumed to be 0.4. Consistent with the idea that correlations between different equity markets may increase

[35] It should be emphasized that correlations between asset returns may also decrease during crises. For example, an increase in credit spreads can result in a decrease of the correlation between the returns on Treasury and corporate bonds. In particular, the increase in credit spreads of 1998 led Long Term Capital Management to suffer very large losses; see Edwards (1999) and Jorion (2007, pp. 349–353). For an examination of the effect of overestimating correlations on portfolio risk in the absence of stress testing constraints, see Jorion (2007, pp. 568–570).

[36] For an examination of the portfolio selection implications of a regime-switching model where there is an increase in volatilities and correlations during bear markets, see Ang and Bekaert (2002).

[37] Consider two assets that have different payoffs in two different states. By construction, the correlation coefficient between the payoffs of these assets conditional on one of these states occurring is either -1 or 1. Hence, in order to allow correlation coefficients between asset payoffs different from -1 and 1, a breakdown in correlations across more than two states is required. Therefore, we consider a breakdown in correlations across three states.

[38] In motivating the use of these states in stress testing, we can alternatively proceed as is Section 3.2. For example, suppose that a state is used in stress testing if the payoff of the market portfolio in that state is below $C = 1.7$. Since the payoffs of the market portfolio in states s_1, s_2, and s_3 are, respectively, 1.487, 1.536, and 1.624, all of these states are used in stress testing. Thus, the constrained agent faces three stress testing constraints.

[39] For an examination of the biases arising from making inferences about a breakdown in correlations from estimated conditional correlation coefficients, see, for example, Boyer et al. (1999), Longin and Solnik (2001), and Forbes and Rigobon (2002).

Table 7
Effect of stress testing constraints on security expected returns when there is a breakdown in correlations
(a) Correlation coefficients between asset payoffs conditional on the states used in stress testing

j	1	2	3	4
1	1.00	0.98	0.85	0.94
2		1.00	0.72	0.84
3			1.00	0.98
4				1.00

(b) Security expected returns

j	1	2	3	4
$\beta_j \overline{R}_m$ (CAPM expected return)	13.03	10.56	10.47	11.63
$-(R_{js_1^*} - \beta_j R_{ms_1^*})\delta_{s_1^*}$	0.24	−1.30	0.02	0.58
$-(R_{js_2^*} - \beta_j R_{ms_2^*})\delta_{s_2^*}$	0.02	−0.64	−0.03	0.33
$-(R_{js_3^*} - \beta_j R_{ms_3^*})\delta_{s_3^*}$	0.04	−0.11	−0.10	0.12
Idiosyncratic return adjustment	0.30	−2.05	−0.11	1.03
$\overline{R}_j$	13.33	8.51	10.36	12.66

Note. This table examines the effect of stress testing constraints when there is a breakdown in correlations. The breakdown in correlations occurs in states s_1, s_2, and s_3. Accordingly, there are three stress testing constraints that use these states. The bounds are $T_{s_1} = 8\%$, $T_{s_2} = 6\%$, and $T_{s_3} = 4\%$. Since the three constraints bind for the constrained agent, we have $s_1^* = s_1$, $s_2^* = s_2$, and $s_3^* = s_3$. The rest of the notation is defined as in Table 4. Panel (a) shows the correlation coefficients between asset payoffs conditional on the states used in stress testing. Panel (b) presents the security expected returns. With the exception of the correlation coefficients, all numbers are annualized and reported in percentage points.

during crises, the panel indicates that the conditional correlation coefficients are larger than the unconditional correlation coefficients.[40]

Panel (b) of Table 1 shows that the payoffs of all risky securities in state s_1 are smaller than those in state s_2, which in turn are smaller than those in state s_3. Thus, we examine stress testing constraints where the maximum loss that the constrained agent can suffer in state s_1 is larger than that he or she can suffer in state s_2, which in turn is larger than that he or she can suffer in state s_3. Specifically, we assume that the bounds are $T_{s_1} = 8\%$, $T_{s_2} = 6\%$, and $T_{s_3} = 4\%$. Since the three constraints bind for the constrained agent, we have $s_1^* = s_1$, $s_2^* = s_2$, and $s_3^* = s_3$.

Panel (b) of Table 7 presents equilibrium security expected returns. Comparing the case of three stress testing constraints with the case of two stress testing constraints with bounds $T_{s_1^*} = 8\%$ and $T_{s_2^*} = 6\%$ as shown in Table 5(b), three results can be seen. First, the CAPM expected returns of all risky securities in Table 7(b) are slightly larger than those in Table 5(b). Second, the idiosyncratic return adjustments in Table 7(b) are within 8 basis points of those in Table 5(b). Third, the security expected returns in Table 7(b) are also within 8 basis points of those in Table 5(b). In sum, as in the case when only two stress testing constraints are imposed, the effect of the three constraints on the expected returns of some securities can be large. However, the effect can also be relatively small for other securities.

[40] The values of these correlation coefficients result from our assumptions on the payoffs of the risky assets in states s_1, s_2, and s_3. While these values are only used for illustrative purposes, examples with similar results can be constructed where the values of the correlation coefficients differ from those in Table 7(a).

4. Conclusion

Over the past twenty years, some investors have suffered sizeable losses due to extreme events (e.g., the crash in the U.S. stock market of 1987 and the terrorist attacks in the U.S. of 2001). Stress testing is now commonly utilized as a tool to prevent such losses. For example, the Committee on the Global Financial System (2005, pp. 1, 15) of the Bank for International Settlements and Scholes (2000) note that financial intermediaries often use stress testing to set risk exposure limits. Accordingly, our paper explores the portfolio selection and asset pricing implications arising from the use of stress testing constraints in a model where one of the agents faces these constraints and the other does not.

We obtain three main results. First, the constrained agent's optimal portfolio exhibits (K^*+2)-*fund separation*, where K^* is the number of binding stress testing constraints. This result simplifies stress testing-constrained portfolio selection problems into (K^*+2)-fund allocation exercises that can be solved in two steps: (1) identifying the K^*+2 funds, and (2) allocating wealth among them. While two of these funds are also used by the unconstrained agent, the remaining K^* funds capture the need to satisfy the constraints. Second, the effect of the constraints on the optimal portfolio is identical to that of an adjustment in the expected payoffs of the risky securities that tends to lower them. As a result, the constraints tend to lead to an increase in the optimal holding of the risk-free security, thereby reducing an agent's exposure to risky securities. Finally, a security's equilibrium expected return is affected by *both* its systematic risk (i.e., beta) and its idiosyncratic returns in the states where the constraints bind. Specifically, securities with negative (positive) idiosyncratic returns in these states have relatively high (low) expected returns in equilibrium. This result provides further motivation to the literature in which security prices are not solely driven by systematic risk.

Acknowledgments

This paper has benefited from the valuable comments of two anonymous referees, Luca Benzoni, Ravi Jagannathan, S. Viswanathan, and seminar participants at the 2006 Financial Management Association Meeting in Salt Lake City, 2006 Portuguese Finance Network Conference in Porto, and 2006 Global Finance Conference in Rio de Janeiro. Two earlier versions of this paper circulated under the titles "Portfolio Selection, Asset Pricing, and Stress Testing" and "Risk Management with Stress Testing: Implications for Portfolio Selection and Asset Pricing." The latter version won the 2006 Financial Management Association Best Paper Award in Risk Management. Baptista gratefully acknowledges research support from the School of Business at The George Washington University.

Appendix A

Proof that portfolios on the mean-variance boundary satisfy Eq. (3). Portfolio $\boldsymbol{q}^*$ is on the mean-variance boundary if and only if it solves

$$\min_{\boldsymbol{q}\in\mathbb{R}^{J+1}} \frac{1}{2}\widehat{\boldsymbol{q}}^\top \boldsymbol{V}\widehat{\boldsymbol{q}} \tag{15}$$

$$\text{s.t.} \quad \widehat{\boldsymbol{q}}^\top \overline{\boldsymbol{d}} + q_{J+1} = \overline{d}^*, \tag{16}$$

$$\widehat{\boldsymbol{q}}^\top \boldsymbol{p} + q_{J+1} = p^* \tag{17}$$

for some $(\bar{d}^*, p^*) \in \mathbb{R} \times \mathbb{R}_{++}$. First-order conditions for $\boldsymbol{q}^*$ to solve problem (15) subject to constraints (16) and (17) are given by

$$\boldsymbol{V}\widehat{\boldsymbol{q}}^* - \lambda_1 \bar{\boldsymbol{d}} - \lambda_2 \boldsymbol{p} = \boldsymbol{0}, \tag{18}$$

$$-\lambda_1 - \lambda_2 = 0, \tag{19}$$

$$(\widehat{\boldsymbol{q}}^*)^\top \bar{\boldsymbol{d}} + q^*_{J+1} = \bar{d}^*, \tag{20}$$

$$(\widehat{\boldsymbol{q}}^*)^\top \boldsymbol{p} + q^*_{J+1} = p^*, \tag{21}$$

where λ_1 and λ_2 are Lagrange multipliers associated with these constraints. Eqs. (18) and (19) imply that

$$\widehat{\boldsymbol{q}}^* = \lambda_1 \boldsymbol{V}^{-1}(\bar{\boldsymbol{d}} - \boldsymbol{p}). \tag{22}$$

Premultiplying Eq. (22) by $\bar{\boldsymbol{d}}^\top$ and using Eq. (20), we have

$$\bar{d}^* - q^*_{J+1} = \lambda_1 \bar{\boldsymbol{d}}^\top \boldsymbol{V}^{-1}(\bar{\boldsymbol{d}} - \boldsymbol{p}). \tag{23}$$

Similarly, premultiplying Eq. (22) by $\boldsymbol{p}^\top$ and using Eq. (21), we have

$$p^* - q^*_{J+1} = \lambda_1 \boldsymbol{p}^\top \boldsymbol{V}^{-1}(\bar{\boldsymbol{d}} - \boldsymbol{p}). \tag{24}$$

It follows from Eqs. (23) and (24) that

$$\frac{\bar{d}^* - p^*}{(\bar{\boldsymbol{d}} - \boldsymbol{p})^\top \boldsymbol{V}^{-1}(\bar{\boldsymbol{d}} - \boldsymbol{p})} = \lambda_1. \tag{25}$$

Eqs. (22) and (25) imply that

$$\widehat{\boldsymbol{q}}^* = \frac{(\bar{d}^* - p^*)\boldsymbol{V}^{-1}(\bar{\boldsymbol{d}} - \boldsymbol{p})}{(\bar{\boldsymbol{d}} - \boldsymbol{p})^\top \boldsymbol{V}^{-1}(\bar{\boldsymbol{d}} - \boldsymbol{p})}. \tag{26}$$

The desired result follows from Eq. (26). □

Proof of Theorem 1. Since constraint (5) binds, we have

$$q^*_{J+1u} = \left(\widehat{\boldsymbol{q}}_u - \widehat{\boldsymbol{q}}^*_u\right)^\top \boldsymbol{p} + q_{J+1u}. \tag{27}$$

Hence, $\widehat{\boldsymbol{q}}^*_u$ solves

$$\max_{\widehat{\boldsymbol{q}} \in \mathbb{R}^J} \widehat{\boldsymbol{q}}^\top \bar{\boldsymbol{d}} + (\widehat{\boldsymbol{q}}_u - \widehat{\boldsymbol{q}})^\top \boldsymbol{p} + q_{J+1u} - \frac{\gamma_u}{2} \widehat{\boldsymbol{q}}^\top \boldsymbol{V} \widehat{\boldsymbol{q}}. \tag{28}$$

A first-order condition for $\widehat{\boldsymbol{q}}^*_u$ to solve maximization problem (28) is given by

$$\bar{\boldsymbol{d}} - \boldsymbol{p} - \gamma_u \boldsymbol{V} \widehat{\boldsymbol{q}}^*_u = \boldsymbol{0}. \tag{29}$$

Eq. (29) implies that

$$\widehat{\boldsymbol{q}}^*_u = \frac{\boldsymbol{V}^{-1}(\bar{\boldsymbol{d}} - \boldsymbol{p})}{\gamma_u}. \tag{30}$$

The desired result follows from Eqs. (27) and (30). □

Proof of Theorem 2. Since constraint (8) binds, we have

$$q^*_{J+1c} = \left(\widehat{\boldsymbol{q}}_c - \widehat{\boldsymbol{q}}^*_c\right)^\top \boldsymbol{p} + q_{J+1c}. \tag{31}$$

Suppose that constraint (9) binds for states $s^*_1, \dots, s^*_{K^*}$, where $K^* \leqslant K$. Then,

$$\left(\widehat{\boldsymbol{q}}^*_c\right)^\top \boldsymbol{d}_s + \left(\widehat{\boldsymbol{q}}_c - \widehat{\boldsymbol{q}}^*_c\right)^\top \boldsymbol{p} + q_{J+1c} = (1 - T_s)\left(\widehat{\boldsymbol{q}}_c^\top \boldsymbol{p} + q_{J+1c}\right), \quad s = s^*_1, \dots, s^*_{K^*},$$

or equivalently,

$$\left(\widehat{\boldsymbol{q}}^*_c\right)^\top (\boldsymbol{d}_s - \boldsymbol{p}) = -T_s\left(\widehat{\boldsymbol{q}}_c^\top \boldsymbol{p} + q_{J+1c}\right), \quad s = s^*_1, \dots, s^*_{K^*}.$$

Hence, $\widehat{\boldsymbol{q}}^*_c$ solves

$$\max_{\widehat{\boldsymbol{q}} \in \mathbb{R}^J} \widehat{\boldsymbol{q}}^\top \bar{\boldsymbol{d}} + (\widehat{\boldsymbol{q}}_c - \widehat{\boldsymbol{q}})^\top \boldsymbol{p} + q_{J+1c} - \frac{\gamma_c}{2}\widehat{\boldsymbol{q}}^\top \boldsymbol{V}\widehat{\boldsymbol{q}} \tag{32}$$

$$\text{s.t.} \quad \widehat{\boldsymbol{q}}^\top (\boldsymbol{d}_s - \boldsymbol{p}) = -T_s\left(\widehat{\boldsymbol{q}}_c^\top \boldsymbol{p} + q_{J+1c}\right), \quad s = s^*_1, \dots, s^*_{K^*}. \tag{33}$$

A first-order condition for $\widehat{\boldsymbol{q}}^*_c$ to solve maximization problem (32) subject to constraints (33) are given by

$$\bar{\boldsymbol{d}} - \boldsymbol{p} - \gamma_c \boldsymbol{V}\widehat{\boldsymbol{q}}^*_c + \sum_{k=1}^{K^*} \lambda_{s^*_k}(\boldsymbol{d}_{s^*_k} - \boldsymbol{p}) = \boldsymbol{0}, \tag{34}$$

$$\left(\widehat{\boldsymbol{q}}^*_c\right)^\top (\boldsymbol{d}_s - \boldsymbol{p}) = -T_s\left(\widehat{\boldsymbol{q}}_c^\top \boldsymbol{p} + q_{J+1c}\right), \quad s = s^*_1, \dots, s^*_{K^*}, \tag{35}$$

where $\lambda_{s^*_1}, \dots, \lambda_{s^*_{K^*}}$ are Lagrange multipliers associated with these constraints. Eq. (34) implies that

$$\widehat{\boldsymbol{q}}^*_c = \frac{\boldsymbol{V}^{-1}[\bar{\boldsymbol{d}} - \boldsymbol{p} + \sum_{k=1}^{K^*} \lambda_{s^*_k}(\boldsymbol{d}_{s^*_k} - \boldsymbol{p})]}{\gamma_c}. \tag{36}$$

The desired result follows from Eqs. (31) and (36). The multipliers can be found as follows. Premultiplying Eq. (36) by $(\boldsymbol{d}_s - \boldsymbol{p})^\top$ and using Eq. (35), we have

$$-T_s\left(\widehat{\boldsymbol{q}}_c^\top \boldsymbol{p} + q_{J+1c}\right) = \frac{(\boldsymbol{d}_s - \boldsymbol{p})^\top \boldsymbol{V}^{-1}[\bar{\boldsymbol{d}} - \boldsymbol{p} + \sum_{k=1}^{K^*} \lambda_{s^*_k}(\boldsymbol{d}_{s^*_k} - \boldsymbol{p})]}{\gamma_c}, \quad s = s^*_1, \dots, s^*_{K^*}. \tag{37}$$

Let $I_s \equiv (\boldsymbol{d}_s - \boldsymbol{p})^\top \boldsymbol{V}^{-1}(\boldsymbol{d} - \boldsymbol{p})$ and $I_{ss^*_k} \equiv (\boldsymbol{d}_s - \boldsymbol{p})^\top \boldsymbol{V}^{-1}(\boldsymbol{d}_{s^*_k} - \boldsymbol{p})$. It follows from Eq. (37) that

$$\sum_{k=1}^{K^*} \lambda_{s^*_k} I_{ss^*_k} = -\left[I_s + T_s\left(\widehat{\boldsymbol{q}}_c^\top \boldsymbol{p} + q_{J+1c}\right)\gamma_c\right], \quad s = s^*_1, \dots, s^*_{K^*}. \tag{38}$$

Let

$$\boldsymbol{X} \equiv [\,\lambda_{s^*_1} \quad \cdots \quad \lambda_{s^*_{K^*}}\,]^\top, \qquad \boldsymbol{U} \equiv [\,\boldsymbol{d}_{s^*_1} - \boldsymbol{p} \quad \cdots \quad \boldsymbol{d}_{s^*_{K^*}} - \boldsymbol{p}\,], \qquad \boldsymbol{Y} \equiv \boldsymbol{U}^\top \boldsymbol{V}^{-1}\boldsymbol{U},$$

and

$$\boldsymbol{Z} \equiv \left[\,-\left[I_{s^*_1} + T_{s^*_1}\left(\widehat{\boldsymbol{q}}_c^\top \boldsymbol{p} + q_{J+1c}\right)\gamma_c\right] \quad \cdots \quad -\left[I_{s^*_{K^*}} + T_{s^*_{K^*}}\left(\widehat{\boldsymbol{q}}_c^\top \boldsymbol{p} + q_{J+1c}\right)\gamma_c\right]\right]^\top.$$

Using Eq. (38), we have $\boldsymbol{YX} = \boldsymbol{Z}$. Hence, $\boldsymbol{X} = \boldsymbol{Y}^{-1}\boldsymbol{Z}$. $\square$

Proof of Theorem 3. The desired result follows from Eqs. (30) and (36). □

Proof of Theorem 4. We begin by showing that market portfolio $\boldsymbol{q}_m$ is inefficient. Since (1) $\boldsymbol{q}_u^*$ is efficient, (2) $\boldsymbol{q}_c^*$ is inefficient, and (3) $\boldsymbol{q}_m$ is the sum of $\boldsymbol{q}_u^*$ and $\boldsymbol{q}_c^*$, $\boldsymbol{q}_m$ is inefficient.

Next, we show that Eq. (14) holds. Using Eqs. (12), (30), and (36), we have

$$\frac{\boldsymbol{V}^{-1}(\bar{\boldsymbol{d}}-\boldsymbol{p}^*)}{\gamma_u}+\frac{\boldsymbol{V}^{-1}[\bar{\boldsymbol{d}}-\boldsymbol{p}^*+\sum_{k=1}^{K^*}\lambda_{s_k^*}(\boldsymbol{d}_{s_k^*}-\boldsymbol{p}^*)]}{\gamma_c}=\mathbf{1}. \tag{39}$$

Premultiplying Eq. (39) by $\boldsymbol{V}$, we have

$$\frac{\bar{\boldsymbol{d}}-\boldsymbol{p}^*}{\gamma_u}+\frac{\bar{\boldsymbol{d}}-\boldsymbol{p}^*+\sum_{k=1}^{K^*}\lambda_{s_k^*}(\boldsymbol{d}_{s_k^*}-\boldsymbol{p}^*)}{\gamma_c}=\boldsymbol{V}\mathbf{1}. \tag{40}$$

Using Eq. (40), we have

$$\frac{\bar{\boldsymbol{d}}(\frac{1}{\gamma_u}+\frac{1}{\gamma_c})+\frac{\sum_{k=1}^{K^*}\lambda_{s_k^*}\boldsymbol{d}_{s_k^*}}{\gamma_c}-\boldsymbol{V}\mathbf{1}}{\frac{1}{\gamma_u}+\frac{1+\sum_{k=1}^{K^*}\lambda_{s_k^*}}{\gamma_c}}=\boldsymbol{p}^*. \tag{41}$$

It follows from Eq. (41) that

$$\frac{\bar{d}_j(\frac{1}{\gamma_u}+\frac{1}{\gamma_c})+\frac{\sum_{k=1}^{K^*}\lambda_{s_k^*}d_{js_k^*}}{\gamma_c}-V_{jm}}{\frac{1}{\gamma_u}+\frac{1+\sum_{k=1}^{K^*}\lambda_{s_k^*}}{\gamma_c}}=p_j^*, \tag{42}$$

where V_{jm} denotes the covariance between the payoffs of security j and the market portfolio. Using Eq. (42), we have

$$\bar{R}_j\left(\frac{1}{\gamma_u}+\frac{1}{\gamma_c}\right)+\frac{\sum_{k=1}^{K^*}\lambda_{s_k^*}R_{js_k^*}^*}{\gamma_c}=\frac{V_{jm}}{p_j^*}. \tag{43}$$

Eq. (41) implies that

$$\frac{\bar{d}_m(\frac{1}{\gamma_u}+\frac{1}{\gamma_c})+\frac{\sum_{k=1}^{K^*}\lambda_{s_k^*}d_{ms_k^*}}{\gamma_c}-V_{mm}}{\frac{1}{\gamma_u}+\frac{1+\sum_{k=1}^{K^*}\lambda_{s_k^*}}{\gamma_c}}=p_m^*, \tag{44}$$

where $\bar{d}_m$ denotes the expected payoff of the market portfolio, d_{ms} denotes the payoff of the market portfolio in state s, V_{mm} denotes the variance of the market portfolio payoff, and p_m^* denotes the price of the market portfolio. Using Eq. (44), we have

$$\bar{R}_m\left(\frac{1}{\gamma_u}+\frac{1}{\gamma_c}\right)+\frac{\sum_{k=1}^{K^*}\lambda_{s_k^*}R_{ms_k^*}}{\gamma_c}=\frac{V_{mm}}{p_m^*}. \tag{45}$$

The definition of β_j implies that

$$\beta_j=\frac{V_{jm}p_m^*}{V_{mm}p_j^*}. \tag{46}$$

It follows from Eqs. (43), (45), and (46) that

$$\overline{R}_j\left(\frac{1}{\gamma_u}+\frac{1}{\gamma_c}\right)+\frac{\sum_{k=1}^{K^*}\lambda_{s_k^*}R^*_{js_k^*}}{\gamma_c}=\beta_j\left[\overline{R}_m\left(\frac{1}{\gamma_u}+\frac{1}{\gamma_c}\right)+\frac{\sum_{k=1}^{K^*}\lambda_{s_k^*}R_{ms_k^*}}{\gamma_c}\right]. \tag{47}$$

The desired result follows from Eq. (47). □

References

Alexander, G.J., Baptista, A.M., 2002. Economic implications of using a mean-VaR model for portfolio selection: A comparison with mean-variance analysis. J. Econ. Dynam. Control 26, 1159–1193.

Alexander, G.J., Baptista, A.M., 2004. A comparison of VaR and CVaR constraints on portfolio selection with the mean-variance model. Manage. Sci. 50, 1261–1273.

Alexander, G.J., Baptista, A.M., 2006a. Does the Basle Capital Accord reduce bank fragility? An assessment of the value-at-risk approach. J. Monet. Econ. 53, 1631–1660.

Alexander, G.J., Baptista, A.M., 2006b. Portfolio selection with a drawdown constraint. J. Banking Finance 30, 3171–3189.

Alexander, G.J., Baptista, A.M., 2008. Active portfolio management with benchmarking: Adding a value-at-risk constraint. J. Econ. Dynam. Control 32, 779–820.

Ang, A., Bekaert, G., 2002. International asset allocation with regime shifts. Rev. Finan. Stud. 15, 1137–1187.

Ang, A., Chen, J., Xing, Y., 2006a. Downside risk. Rev. Finan. Stud. 19, 1191–1239.

Ang, A., Hodrick, R.J., Xing, Y., Zhang, X., 2006b. The cross-section of volatility and expected returns. J. Finance 51, 259–299.

Ang, A., Hodrick, R.J., Xing, Y., Zhang, X., in press. High idiosyncratic volatility and low returns: International and further U.S. evidence. J. Finan. Econ.

Bali, T.G., Cakici, N., Yan, X., Zhang, Z., 2005. Does idiosyncratic risk really matter? J. Finance 60, 905–929.

Barro, R.J., 2006. Rare disasters and asset markets in the twentieth century. Quart. J. Econ. 121, 823–866.

Basak, S., 1995. A general equilibrium model of portfolio insurance. Rev. Finan. Stud. 8, 1059–1190.

Basak, S., Shapiro, A., 2001. Value-at-risk-based risk management: Optimal policies and asset prices. Rev. Finan. Stud. 14, 371–405.

Basak, S., Shapiro, A., Teplá, L., 2006. Risk management with benchmarking. Manage. Sci. 52, 542–557.

Basak, S., Cass, D., Licari, J.M., Pavlova, A., in press-a. Multiplicity in general financial equilibrium with portfolio constraints. J. Econ. Theory.

Basak, S., Shapiro, A., Pavlova, A., in press-b. Offsetting the implicit incentives: Risk shifting and benefits of benchmarking in money management. J. Banking Finance.

Basle Committee on Banking Supervision, 2005. Amendment to the Capital Accord to incorporate market risks. Available at http://www.bis.org/publ/bcbs119.pdf.

Berk, J.B., 1997. Necessary conditions for the CAPM. J. Econ. Theory 73, 245–257.

Berkowitz, J., O'Brien, J., 2002. How accurate are value-at-risk models at commercial banks? J. Finance 57, 1093–1111.

Black, F., 1972. Capital market equilibrium with restricted borrowing. J. Bus. 45, 444–455.

Black, F., Perold, A.F., 1992. Theory of constant proportion portfolio insurance. J. Econ. Dynam. Control 16, 403–426.

Boyer, B.H., Gibson, M.S., Loretan, M., 1999. Pitfalls in tests for changes in correlations. International Finance Discussion Paper 597, Board of Governors of the Federal Reserve System.

Campbell, J.Y., Lettau, M., Malkiel, B.G., Xu, Y., 2001. Have individual stocks become more volatile? An empirical exploration of idiosyncratic risk. J. Finance 56, 1–43.

Cass, D., Pavlova, A., 2004. On trees and logs. J. Econ. Theory 116, 41–83.

Chakrabarti, R., Roll, R., 2002. East Asia and Europe during the 1997 Asian collapse: A clinical study of a financial crisis. J. Finan. Markets 5, 1–30.

Committee on the Global Financial System, 2005. Stress testing at major financial institutions: Survey results and practice. Available at http://www.bis.org/publ/cgfs24.htm.

Cuoco, D., Liu, H., 2006. An analysis of VaR-based capital requirements. J. Finan. Intermediation 15, 362–394.

Cvitanić, J., Karatzas, I., 1992. Convex duality in constrained portfolio optimization. Ann. Appl. Probab. 2, 767–818.

Cvitanić, J., Karatzas, I., 1995. On portfolio optimization under "drawdown" constraints. IMA Lecture Notes in Math. Appl. 65, 77–88.

Daníelsson, J., Zigrand, J.-P., 2008. Equilibrium asset pricing with systemic risk. Econ. Theory 35, 293–319.

Daníelsson, J., Taylor, A., Zigrand, J.-P., 2005. Highwaymen or heroes: Should hedge funds be regulated? A survey. J. Finan. Stability 1, 522–543.
Dittmar, R.F., 2002. Non-linear pricing kernels, kurtosis preference, and the cross section of equity returns. J. Finance 57, 369–403.
Edwards, F.R., 1999. Hedge funds and the collapse of long-term capital management. J. Econ. Perspect. 13, 189–210.
Forbes, K.J., Rigobon, R., 2002. No contagion, only interdependence: Measuring stock market comovements. J. Finance 57, 2223–2261.
Francis, J.C., 1978. Portfolio analysis of asset and liability management in small-, medium- and large-sized banks. J. Monet. Econ. 4, 459–480.
Goyal, A., Santa-Clara, P., 2003. Idiosyncratic risk matters! J. Finance 58, 975–1007.
Grinold, R.C., Kahn, R.N., 2000. Active Portfolio Management. McGraw–Hill, New York.
Grossman, S.J., Vila, J.-L., 1992. Optimal dynamic trading with leverage constraints. J. Finan. Quant. Anal. 27, 151–168.
Grossman, S.J., Zhou, Z., 1993. Optimal investment strategies for controlling drawdowns. Math. Finance 3, 241–276.
Guo, H., Savickas, R., 2006. Idiosyncratic volatility, stock market volatility, and expected stock returns. J. Business Econ. Statist. 24, 43–56.
Harris, R.G., 1980. A general equilibrium analysis of the capital asset pricing model. J. Finan. Quant. Anal. 15, 99–122.
Hart, O.D., Jaffee, M.D., 1974. On the application of portfolio theory to depository financial intermediaries. Rev. Econ. Stud. 41, 129–147.
Harvey, C.R., Siddique, A., 2000. Conditional skewness in asset pricing tests. J. Finance 55, 1263–1295.
Huang, C., Litzenberger, R.H., 1988. Foundations for Financial Economics. Prentice Hall, Englewood Cliffs, NJ.
Jagannathan, R., Korajczyk, R.A., 1986. Assessing the market timing performance of managed portfolios. J. Bus. 59, 217–235.
Jagannathan, R., Ma, T., 2002. Risk reduction in large portfolios: Why imposing the wrong constraints helps. Working Paper, National Bureau of Econ. Research.
Jorion, P., 2002. How informative are value-at-risk disclosures? Acc. Rev. 77, 911–931.
Jorion, P., 2007. Value at Risk: The New Benchmark for Controlling Market Risk. McGraw–Hill, New York.
Kim, D., Santomero, A.M., 1988. Risk in banking and capital regulation. J. Finance 43, 1219–1233.
King, M.A., Wadhwani, S., 1990. Transmission of volatility between stock markets. Rev. Finan. Stud. 3, 5–33.
Kodres, L.E., Pritsker, M., 2002. A rational expectations model of financial contagion. J. Finance 57, 768–799.
Koehn, M., Santomero, A.M., 1980. Regulation of bank capital and portfolio risk. J. Finance 35, 1235–1244.
Kraus, A., Litzenberger, R.H., 1976. Skewness preference and the valuation of risk assets. J. Finance 31, 1085–1100.
Levy, H., 1978. Equilibrium in an imperfect market. Amer. Econ. Rev. 68, 643–658.
Lintner, J., 1965. The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. Rev. Econ. Statist. 47, 13–37.
Longin, F., 2000. From value at risk to stress testing: The extreme value approach. J. Banking Finance 24, 1097–1130.
Longin, F., Solnik, B., 2001. Extreme correlation of international equity markets. J. Finance 56, 649–676.
Maenhout, P.J., 2004. Robust portfolio rules and asset pricing. Rev. Finan. Stud. 17, 951–983.
Malkiel, B.G., Xu, Y., 2002. Idiosyncratic risk and security returns. Working Paper, University of Texas at Dallas.
Malkiel, B.G., Xu, Y., 2003. Investigating the behavior of idiosyncratic volatility. J. Bus. 76, 613–644.
Markowitz, H.M., 1952. Portfolio selection. J. Finance 7, 77–91.
Markowitz, H.M., 1959. Portfolio Selection: Efficient Diversification of Investments. Yale University Press, New Haven, CT.
Markowitz, H.M., 1987. Mean-Variance Analysis in Portfolio Choice and Capital Markets. Basil Blackwell, New York.
Markowitz, H.M., van Dijk, E.L., 2003. Single-period mean-variance analysis in a changing world. Finan. Analysts J. 59, 30–44.
Mehra, R., Prescott, E.C., 1985. The equity premium: A puzzle. J. Monet. Econ. 15, 145–161.
Merton, R.C., 1981. On market timing and investment performance. I. An equilibrium theory of value for market forecasts. J. Bus. 54, 363–406.
Merton, R.C., 1987. A simple model of capital market equilibrium with incomplete information. J. Finance 42, 483–510.
Mossin, J., 1966. Equilibrium in a capital asset market. Econometrica 34, 768–783.
Rietz, T.A., 1988. The equity risk premium: A solution. J. Monet. Econ. 22, 117–131.
Rochet, J.-C., 1992. Capital requirements and the behaviour of commercial banks. Europ. Econ. Rev. 36, 1137–1178.
Roll, R., 1992. A mean/variance analysis of tracking error. J. Portfol. Manage. 18, 13–22.
Scholes, M.S., 2000. Crisis and risk management. Amer. Econ. Revista 90, 17–21.
Sentana, E., 2003. Mean-variance portfolio allocation with a value at risk constraint. Rev. Econ. Financiera 1, 4–14.

Sharpe, W.F., 1964. Capital asset prices: A theory of market equilibrium under conditions of risk. J. Finance 19, 425–442.
Sharpe, W.F., 1987. Asset Allocation Tools. Scientific Press, Palo Alto, CA.
Sorge, M., 2004. Stress-testing financial systems: An overview of current methodologies. Working Paper, Bank for International Settlements. Available at http://www.bis.org/publ/work165.pdf.
Yiu, K.F.C., 2004. Optimal portfolios under a value-at-risk constraint. J. Econ. Dynam. Control 28, 1317–1334.